An alternative view of denoising diffusion models

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Problem set-up
Sampling with iterative algorithms

- **Sampling from probability distribution** $p(x) = \frac{1}{Z} \exp(-f(x))$
  - high-dimensional and "complex"
  - f given (without Z) or f estimated from i.i.d. data

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- **Applications**
  - Image generation $p(x)$
  - Conditional image generation $p(x|y) \propto p(y|x)p(x)$
  - Protein discovery (Frey et al., 2024), etc.

Application to image generation
"Panda riding a bicycle in Paris"

https://stablediffusionweb.com/

Application to image generation
“Darth vador riding a bicycle in the grand canyon”

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- **Applications**
  - Image generation $p(x)$
  - Conditional image generation $p(x|y) \propto p(y|x)p(x)$
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- **Main difficulty**
  - Multimodal distributions
  - Curse of dimensionality

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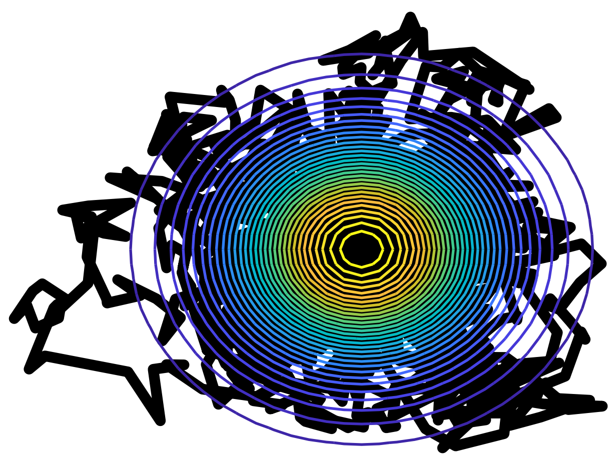
- **Langevin algorithms**
  - Discretization of diffusion $dX_t = -\nabla f(X_t)dt + \sqrt{2}dB_t$:

$$x_{k+1} = x_k - \gamma \nabla f(x_k) + \sqrt{2\gamma} \cdot \mathcal{N}(0, I)$$

  - (slow) convergence (see, e.g., Bakry et al., 2008)
  - fast for smooth log-concave distributions (e.g., f convex) (Dalalyan, 2017, Durmus and Moulines, 2017, Chewi, 2022, etc.)

From log-concave to non-log-concave



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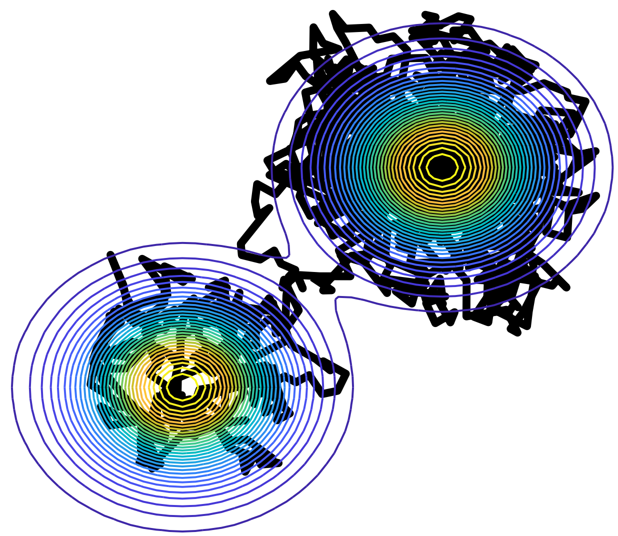

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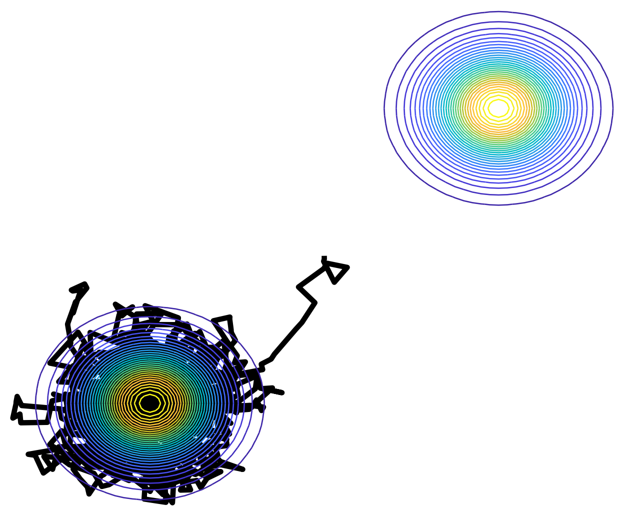

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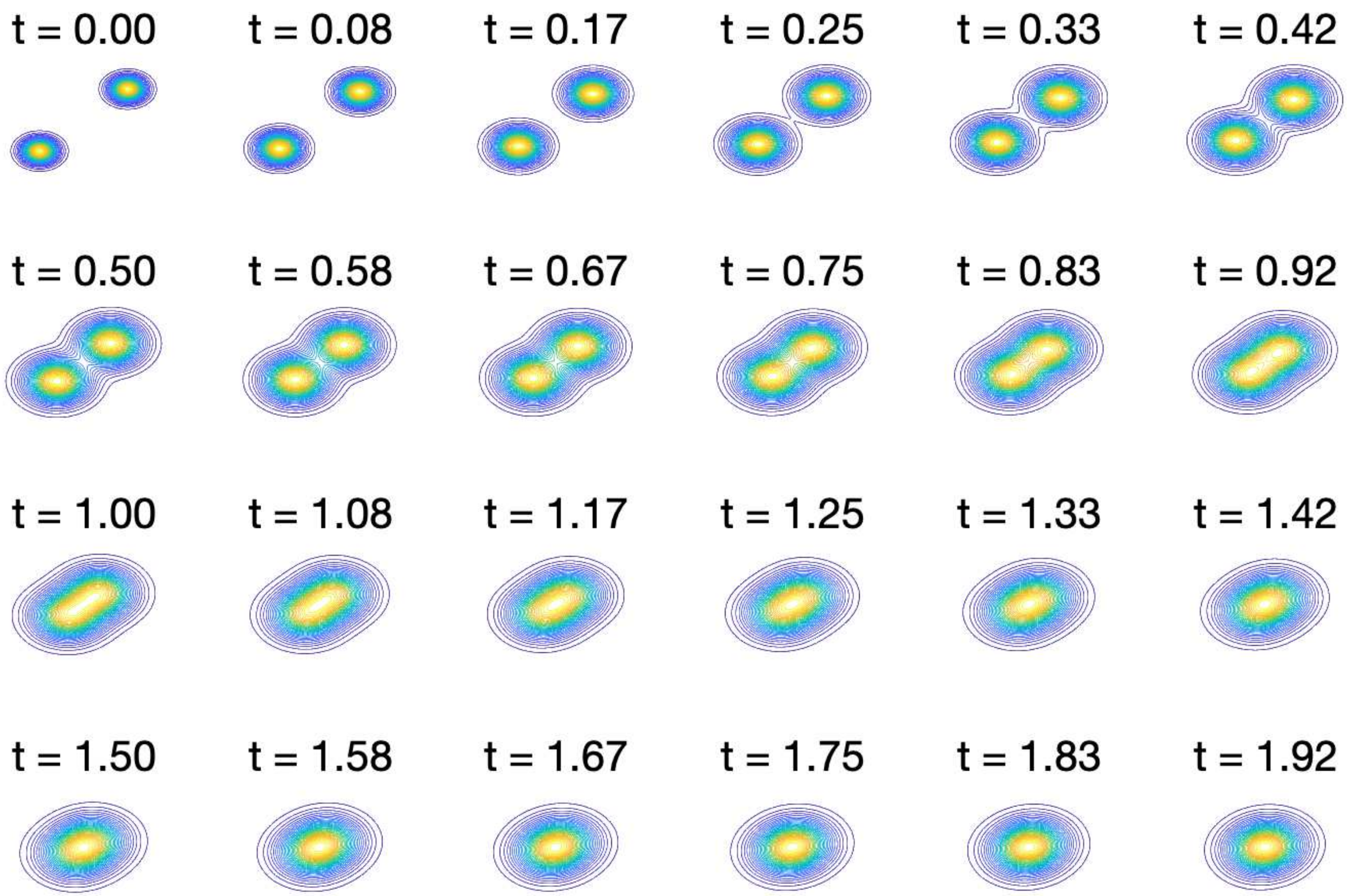
- **Going beyond log-concave distributions**

A short introduction to denoising diffusion models (Song and Ermon, 2019, Song et al., 2019)

[following expositions from Bortoli (2023) and Peyré (2023)]

- **Forward flow**
  - Ornstein-Uhlenbeck process $dX_t = -X_t dt + \sqrt{2} dB_t$
  - started from $p(x) \propto \exp(-f(x))$ at time $t = 0$
  - marginal distribution: $X_t = e^{-t} X_0 + \sqrt{1 - e^{-2t}} \cdot \mathcal{N}(0, I)$
    (explicit integration: $X_t = e^{-t} X_0 + e^{-t} B_{e^{2t}-1}$)

From data to standard Gaussian



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- **Backward flow**
  - For T large, $X_T \approx \mathcal{N}(0, I) \quad \Rightarrow \quad$ backward simulations
  - $Y_t = X_{T-t}$ follows $dY_t = [Y_t + 2\nabla \log r_{T-t}(Y_t)]dt + \sqrt{2} dB_t$ with r_t the density of X_t

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  - Simulate the backward SDE using "only" the densities of X_t

$$y_{k+1} = y_k + \gamma y_k + 2\gamma \nabla \log r_{T-\gamma k}(y_k) + \sqrt{2\gamma} \cdot \mathcal{N}(0, I)$$

Denoising score matching

- **Score functions after adding noise** $\nabla \log r_t(x) = \frac{\nabla r_t}{r_t}(x)$
  - with r_t density of $X_t = e^{-t} X_0 + \sqrt{1 - e^{-2t}} \cdot \mathcal{N}(0, I)$
  - equivalent to density of $X_0 + e^t \sqrt{1 - e^{-2t}} \cdot \mathcal{N}(0, I) = X_0 + \sigma \cdot \mathcal{N}(0, I)$

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- **Empirical Bayes** (Robbins, 1956, Miyasawa, 1961)
  - Notation: q_σ density of $Y = X + \sigma \cdot \mathcal{N}(0, I)$
  - Key result: $\mathbb{E}\big[X|Y\big] = Y + \sigma^2 \nabla \log q_\sigma(Y)$
  - Used within sampling procedure by Saremi and Hyvärinen (2019)
  - Proof by integration by parts
  - No need to know the normalization constant

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- **Denoising score matching** (Hyvärinen, 2005, Vincent, 2011)
  - Estimate the density of the noisy variable y by minimizing

$$\frac{1}{n} \sum_{i=1}^{n} \big\| x_i - y_i - \sigma^2 \nabla \log q_\sigma(y_i | \theta) \big\|^2$$

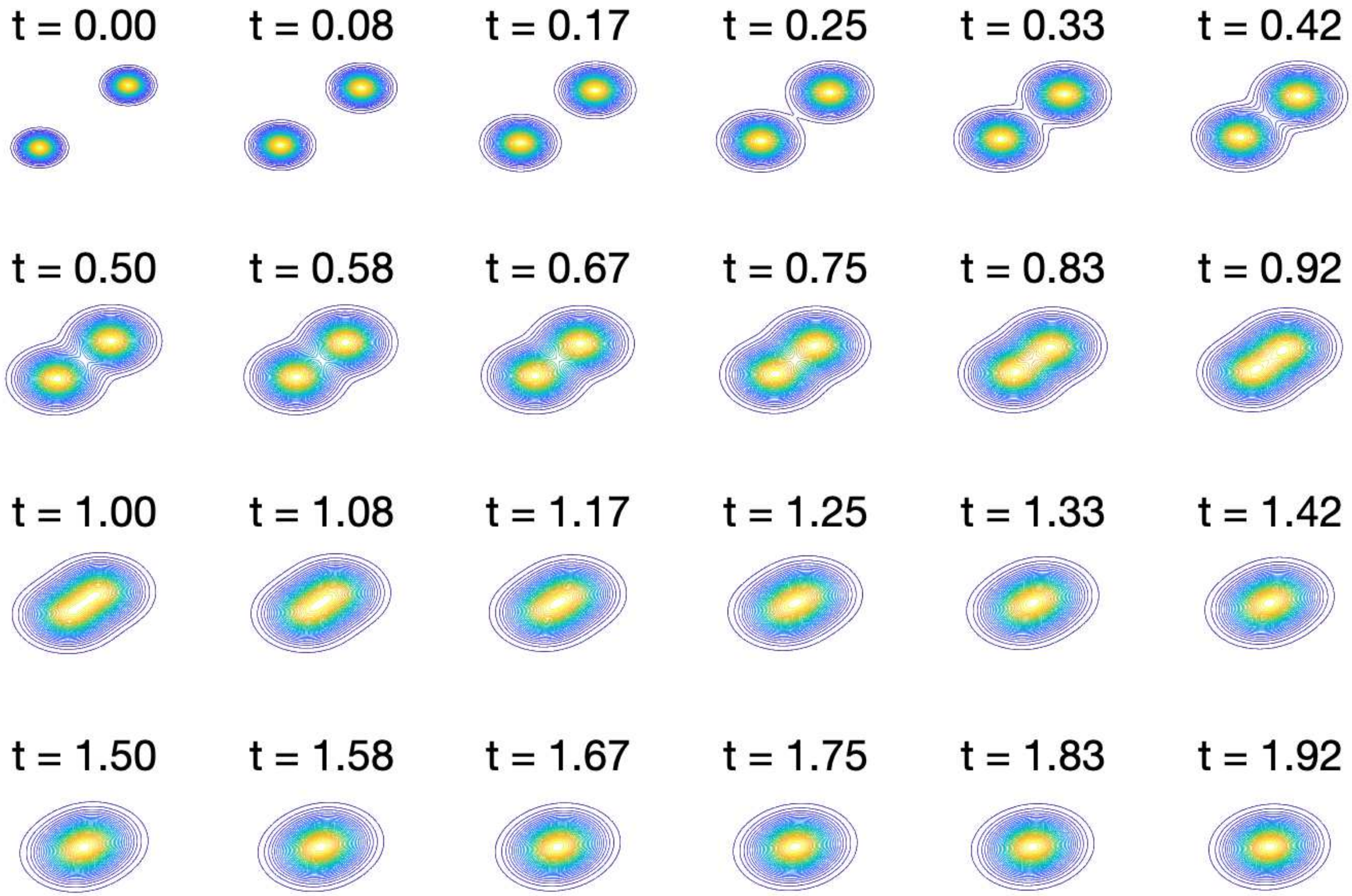
A short introduction to denoising diffusion models (Song and Ermon, 2019, Song et al., 2019)

- **Learning score functions of noisy samples at various scales**
  - Denoising score matching
- **Denoising diffusion models**
  - Start from T large, $y_0 = X_T$, and discretize the backward SDE

$$y_{k+1} = y_k + \gamma y_k + 2\gamma e^{t_k} \nabla \log q_{\sigma_k}(y_k e^{t_k}) + \sqrt{2\gamma} \cdot \mathcal{N}(0, I)$$

  - with $t_k = T - \gamma k$, and $\sigma_k = e^{T-\gamma k}\sqrt{1 - e^{-2T+2\gamma k}}$

From data to standard Gaussian



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- **Alternative view** (Saremi, Park, B., 2023)
  - Diffusion free!

Sampling from a single measurement (Saremi and Hyvärinen, 2019)

- **Algorithm**

1. Learn score at single scale σ: $Y = X + \sigma \cdot \mathcal{N}(0, I)$
2. Sample Y using Langevin diffusions ("walk")
3. Denoise Y ("jump")

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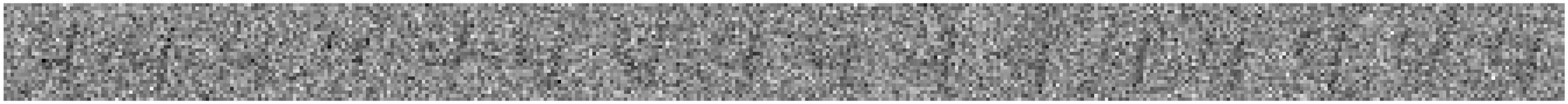
- **Comparison to diffusions**
  - More stable, easier to run (single hyperparameter)
  - σ is too large: Denoising is too "fuzzy"
  - σ is too small: Sampling is difficult

Sampling from a single measurement (Saremi and Hyvärinen, 2019)

Noisy digits (σ=1)



Walk-jump sampling (σ=1)

Walk-jump sampling (σ=1/2)

Sampling from a single measurement (Saremi, Srivastava, B., 2023)

Empirical Bayes with multiple measurements

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- **Multiple measurements:** $Y_i = X + \varepsilon_i$, $i = 1, \ldots, m$
  - Posterior mean: $\mathbb{E}\big[X|Y_1, \ldots, Y_m\big] = \bar{Y}_{1:m} + \frac{\sigma^2}{m} \nabla \log q_{\sigma/\sqrt{m}}(\bar{Y}_{1:m})$
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  - Increased concentration around the mean (S., P. and B., 2023)

$$W_2\big(\text{law of } X, \text{law of } \mathbb{E}[X|Y_1, \ldots, Y_m]\big)^2 \leqslant \frac{\sigma^2 d}{m}$$

  - Improved results with "strong" priors

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- **Idea #1** (Saremi and Srivastava, 2022)
  - Sampling X by sampling $Y_1, \ldots, Y_m$ and then Empirical Bayes

Multimeasurement generative models (Saremi and Srivastava, 2022)

x y_1 y_2 y_3 y_4 $\bar{y}_{1:m}$

$\mathbb{E}[x|y_1, \ldots, y_m]$

- **Still hard to sample from** $(y_1, \ldots, y_m)$

Idea #2: Sequential denoising (S., P. and B., 2023)

- **Multiple measurements:** $Y_i = X + \varepsilon_i$, $i = 1, \ldots, m$
- **Algorithm**
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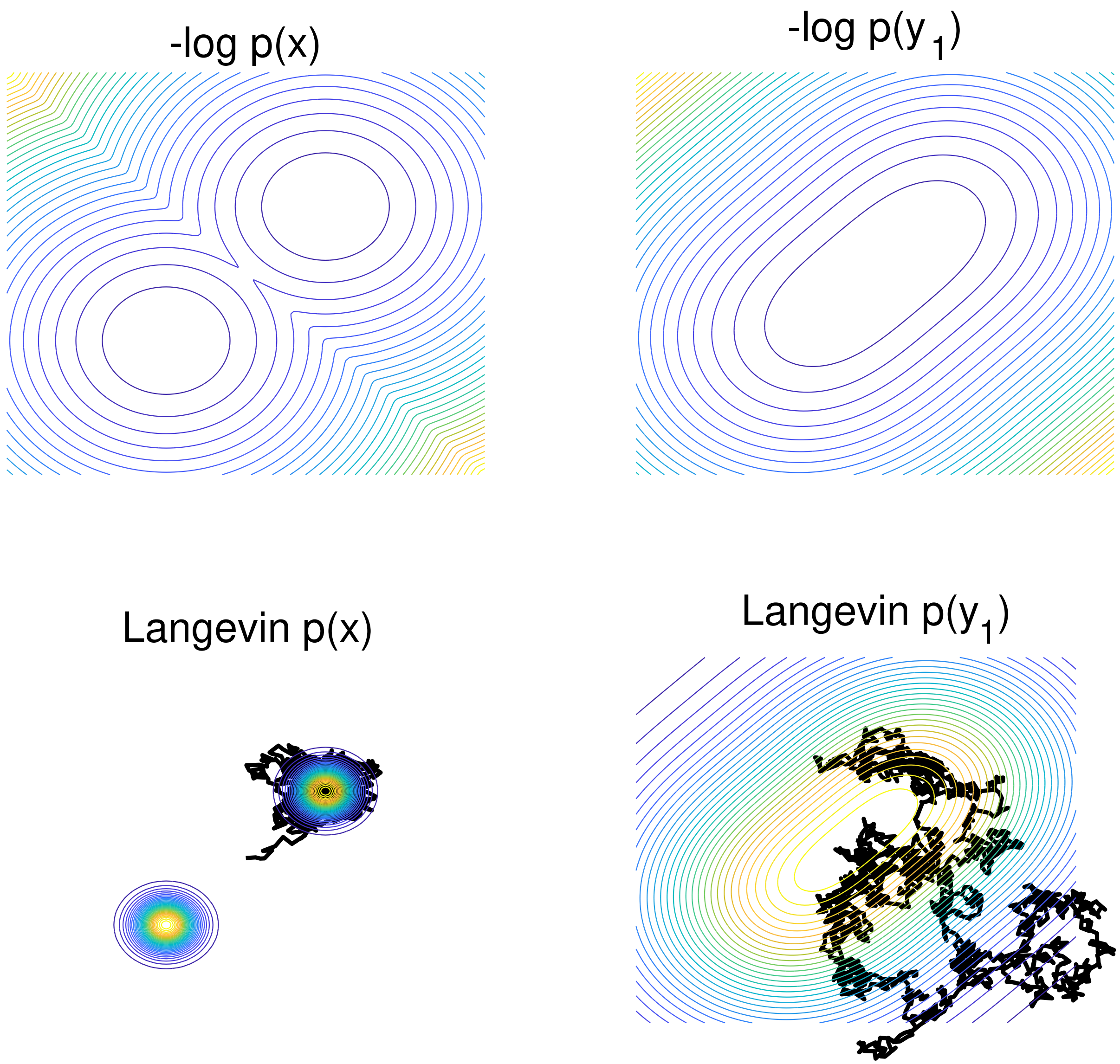
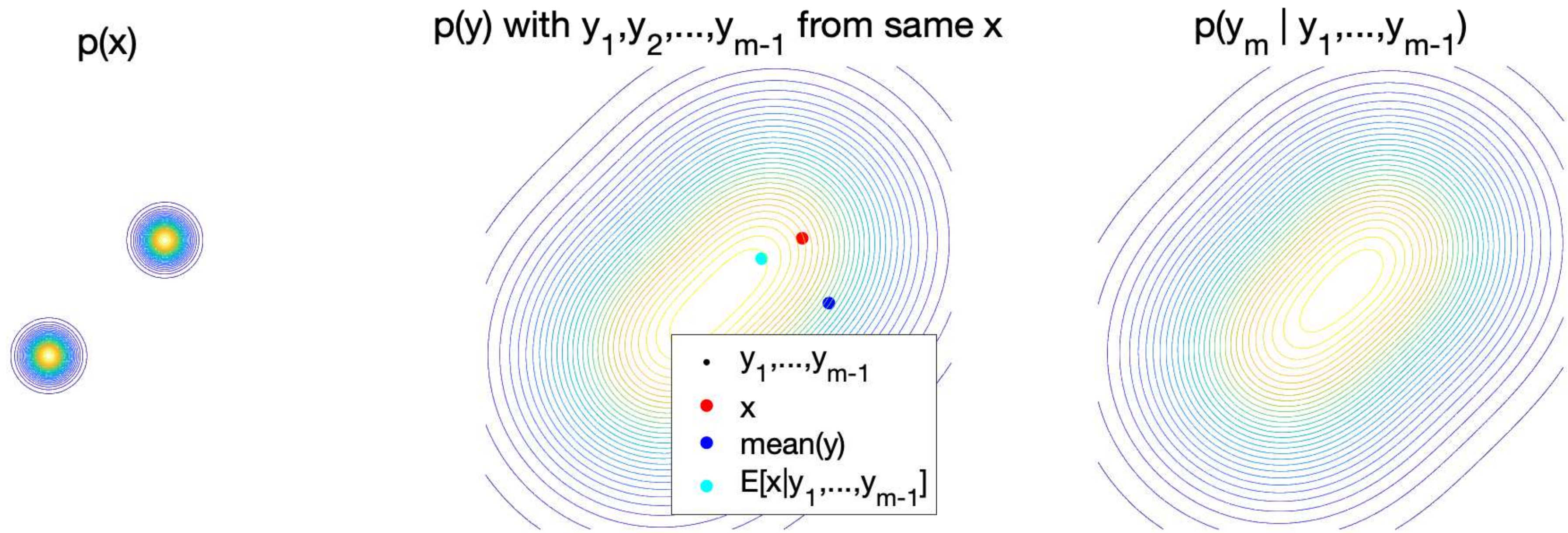
- **Sampling steps using Langevin algorithms**
  - Overall non-Markovian
  - Each sampling step Markovian

First step

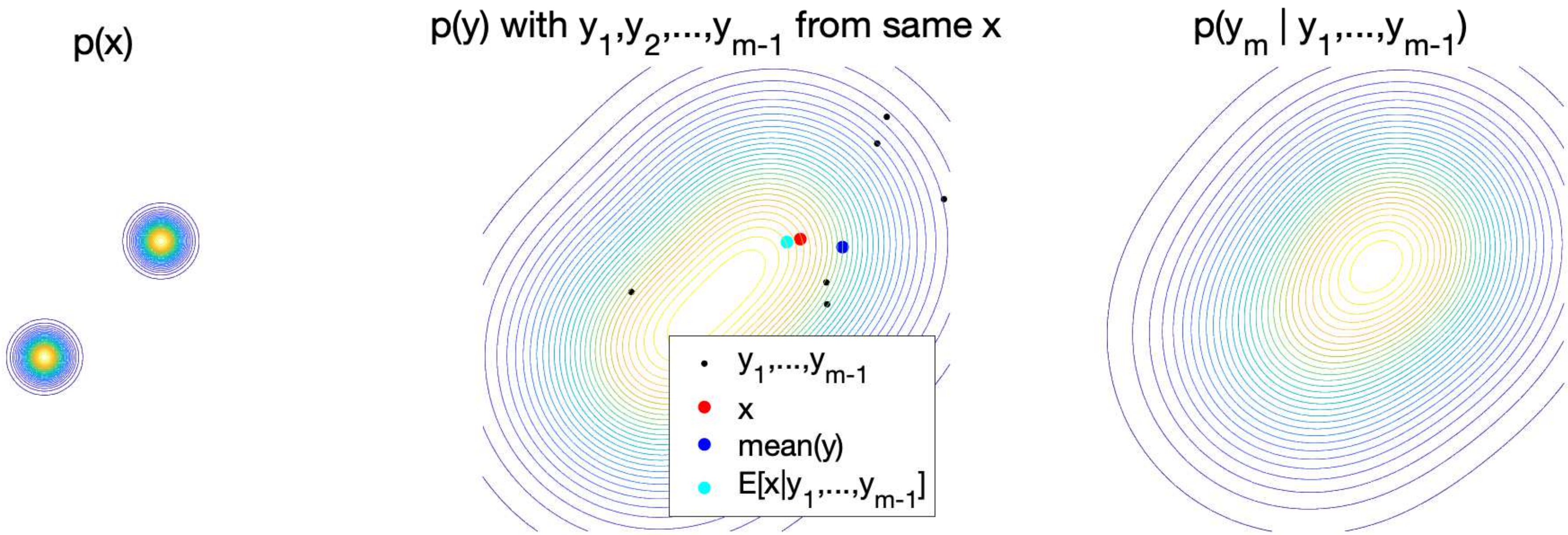


$m = 2$



p(x)
p(y) with y1,y2,...,ym-1 from same x
p(ym | y1,...,ym-1)
y1,...,ym-1
x
mean(y)
E[x|y1,...,ym-1]

$m = 8$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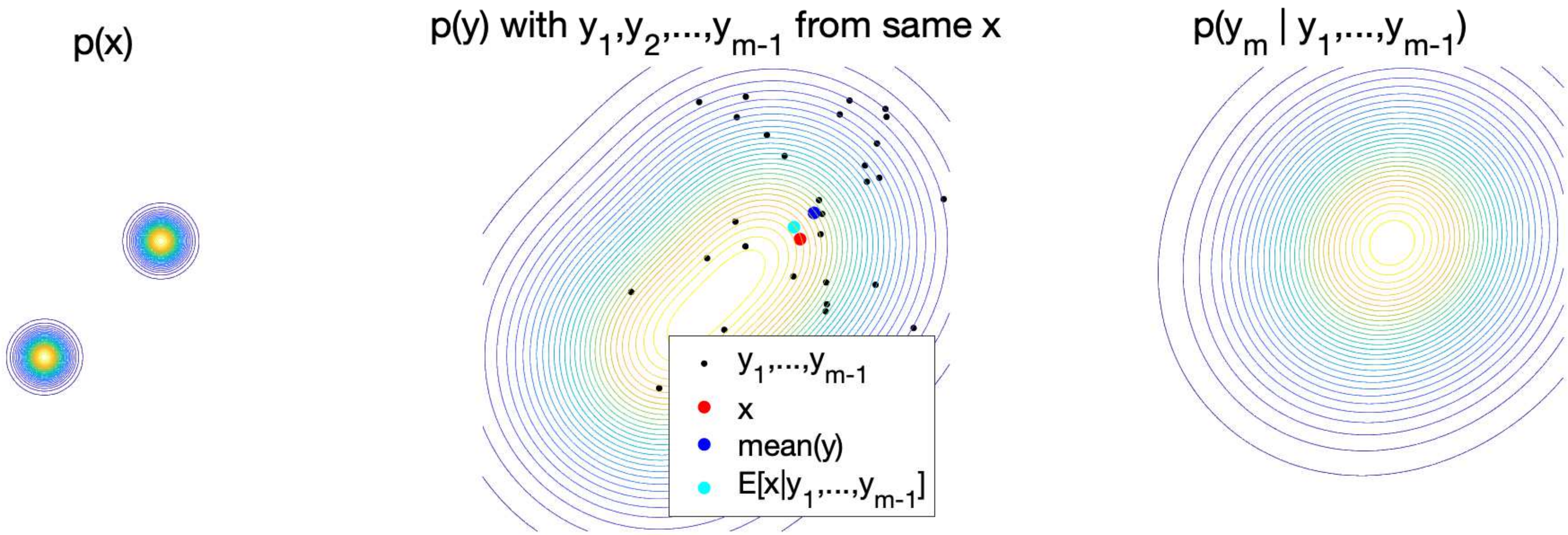
$m = 32$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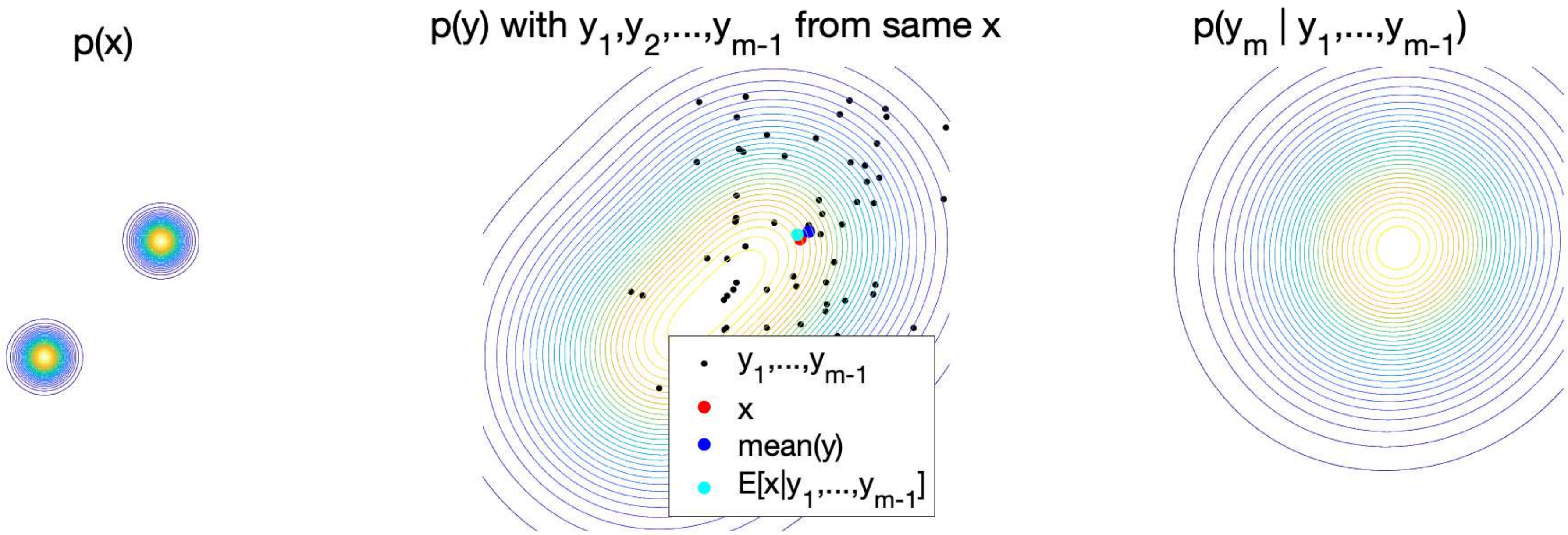

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- **Sampling steps using Langevin algorithms**
  - Feasibility:

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- **Main benefit**
  - If σ large enough, only log-concave distributions to sample from
  - If m large enough, $\frac{\sigma}{\sqrt{m}}$ is small enough to obtain clean samples

More and more log-concave

- **Single measurement:** $Y = X + \sigma \cdot \mathcal{N}(0, I)$
  - Enough Gaussian blurring leads to unimodality (Loog et al., 2001)
  - Enough Gaussian blurring leads to log-concavity

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  - "Proof" (see paper for quantitative statements)

$$\nabla^2 \log q(y) = -\frac{1}{\sigma^2}\Big[I - \frac{1}{\sigma^2}\mathrm{cov}(X|Y = y)\Big]$$

(e.g., for Gaussian mixtures: if $\sigma^2 \geqslant$ diameter of means)

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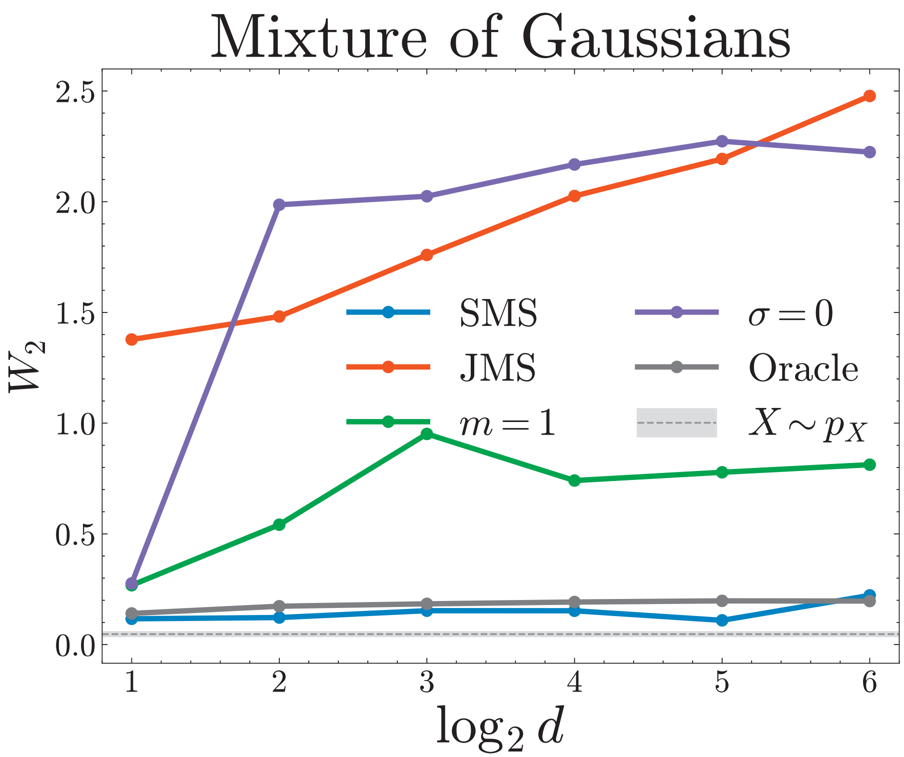
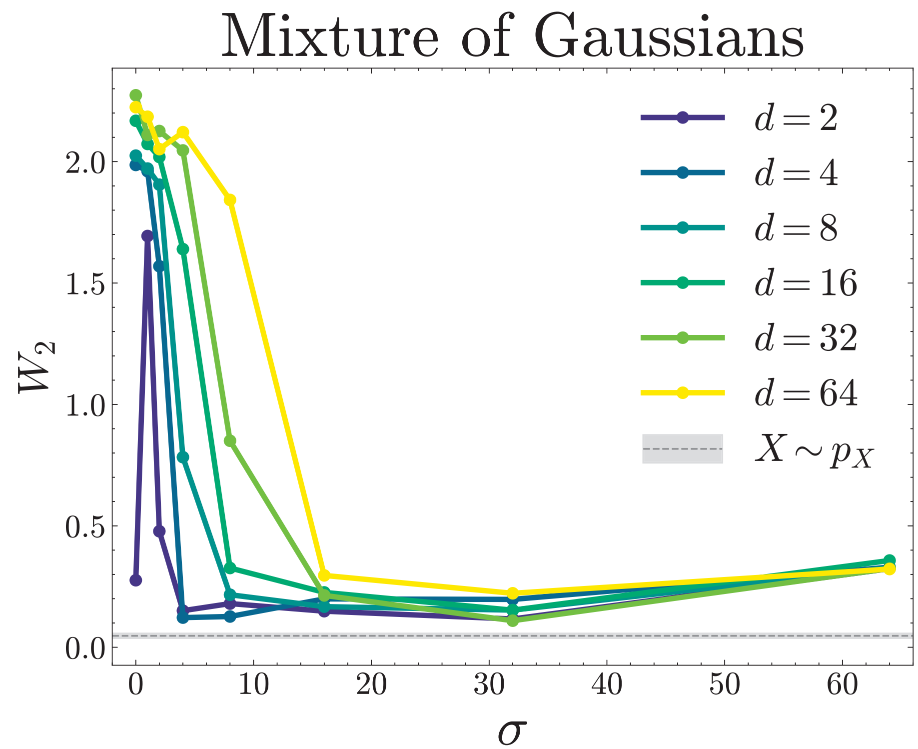
  - Conditioning reduces uncertainty (on average)
  - See precise statements in paper for Gaussian mixtures

Synthetic experiments

- **Mixtures of two Gaussians**
  - covariance matrices $\tau^2 I$, $\Delta\mu = 6 \cdot (1, \dots, 1) \in \mathbb{R}^d$





  - SMS (sequential multimeasurement sampling)
  - JMS (joint multimeasurement sampling)

Discussion

- **Sampling from score functions of smoothed densities**
  - Similar steps to denoising diffusion models
  - Clear initialization: σ large enough to obtain log-concavity
  - m large enough to obtain good quality samples
  - Two hyperparameters: noise σ and number of measurements m

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- **Extensions**
  - Application to image generation
  - Link with stochastic localization (Montanari, 2023)
  - Theoretical analysis of running time (see Chen et al., 2023)
  - Beyond Gaussians and Euclidean geometry
  - Conditional sampling
  - Rigorous empirical evaluation

Link with stochastic localization (Montanari, 2023)

- Define $Z_t = tX + B_t$ with X data and B_t Brownian motion
  - Fact 1: Marginal distribution of $\frac{1}{t}Z_t = X + \mathcal{N}(0, \frac{1}{t}I)$

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  - Fact 2: $dZ_t = \mathbb{E}[X|Z_t]dt + dB_t$

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  - Fact 1: Marginal distribution of $\frac{1}{t}Z_t = X + \mathcal{N}(0, \frac{1}{t}I)$
  - Fact 2: $dZ_t = \mathbb{E}[X|Z_t]dt + dB_t$
  - Fact 3: $\mathbb{E}[X|Z_t] = \frac{1}{t}Z_t + \frac{1}{t}\nabla \log q_{1/\sqrt{t}}\big(\frac{1}{t}Z_t\big)$
  - Sample Z_t for t large by discretizing the diffusion

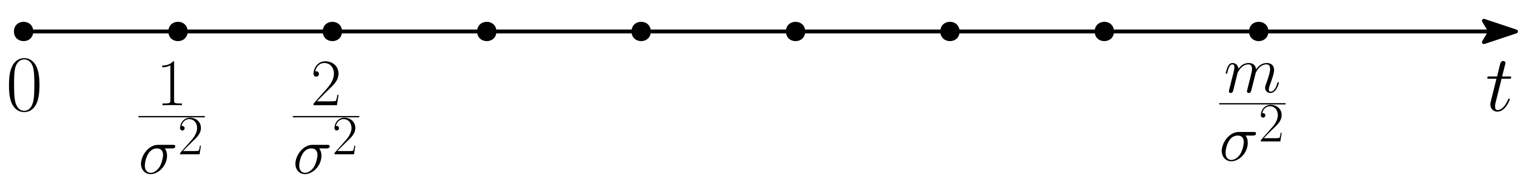
Link with stochastic localization (Montanari, 2023)

- Define $Z_t = tX + B_t$ with X data and B_t Brownian motion
  - Fact 1: Marginal distribution of $\frac{1}{t}Z_t = X + \mathcal{N}(0, \frac{1}{t}I)$
  - Fact 2: $dZ_t = \mathbb{E}[X|Z_t]dt + dB_t$
  - Fact 3: $\mathbb{E}[X|Z_t] = \frac{1}{t}Z_t + \frac{1}{t}\nabla \log q_{1/\sqrt{t}}\big(\frac{1}{t}Z_t\big)$
  - Sample Z_t for t large by discretizing the diffusion

- Define $Y_k = \dfrac{Z_{k\delta} - Z_{(k-1)\delta}}{\delta} = X + \dfrac{B_{k\delta} - B_{(k-1)\delta}}{\delta}$
  - Brownian motion has independent increments

$$\frac{B_{k\delta} - B_{(k-1)\delta}}{\delta} \sim \mathcal{N}(0, \delta^{-1}I)$$

  - Recover multiple measurements with $\delta = \frac{1}{\sigma^2}$



References

- Valentin Bortoli. Generative modeling, https://vdeborto.github.io/project/generative_modeling/, 2023.

- Gabriel Peyré. Denoising Diffusion Models, https://mathematical-tours.github.io/book-sources/optim-ml/OptimML-DiffusionModels.pdf, 2023.

- Bakry, D., Cattiaux, P. and Guillin, A. Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. J. Funct. Anal. 254 727–759, 2008.

- Chen, S., Chewi, S., Li, J., Li, Y., Salim, A., & Zhang, A. R. Sampling is as easy as learning the score: theory for diffusion models with minimal data assumptions. arXiv preprint arXiv:2209.11215. 2022.

- Arnak S. Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. Journal of the Royal Statistical Society. Series B (Statistical Methodology), pp. 651–676, 2017.

- Alain Durmus and Éric Moulines. Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. The Annals of Applied Probability, 27(3):1551 – 1587, 2017.

- Frey, N. C., Berenberg, D., Zadorozhny, K., Kleinhenz, J., Lafrance-Vanasse, J., Hotzel, I., . . . , & Saremi, S. Protein discovery with discrete walk-jump sampling. arXiv preprint arXiv:2306.12360, 2023

- Sinho Chewi. Log-Concave Sampling `https://chewisinho.github.io/main.pdf`, 2023.

- Andrea Montanari. Sampling, Diffusions, and Stochastic Localization. arXiv preprint arXiv:2305.10690, 2023.

- Herbert Robbins. An empirical Bayes approach to statistics. In Proc. Third Berkeley Symp., volume 1, pp. 157–163, 1956.

- Koichi Miyasawa. An empirical Bayes estimator of the mean of a normal population. Bulletin of the International Statistical Institute, 38(4):181–188, 1961.

- Aapo Hyvärinen. Estimation of non-normalized statistical models by score matching. Journal of Machine Learning Research, 6(Apr):695–709, 2005.

- Yang Song and Stefano Ermon. Generative modeling by estimating gradients of the data distribution. Advances in neural information processing systems, 32, 2019.

- Yang Song, Jascha Sohl-Dickstein, Diederik P Kingma, Abhishek Kumar, Stefano Ermon, and Ben Poole. Score-based generative modeling through stochastic differential equations. In International Conference on Learning Representations, 2021.

- Pascal Vincent. A connection between score matching and denoising autoencoders. Neural computation, 23(7):1661–1674, 2011.

- Saeed Saremi and Rupesh Kumar Srivastava. Multimeasurement generative models. In International Conference on Learning Representations, 2022.

- Conforti, Giovanni, Alain Durmus, and Marta Gentiloni Silveri. Score diffusion models without early stopping: finite Fisher information is all you need. arXiv preprint arXiv:2308.12240, 2023

- S. Saremi, R. K. Srivastava, F. Bach. Universal Smoothed Score Functions for Generative Modeling. Technical report, arXiv:2303.11669, 2023.

- S. Saremi, J.-W. Park, F. Bach. Chain of Log-Concave Markov Chains. International Conference on Learning Representations (ICLR), 2024.