ON REDUCTION-BASED PROCESS SEMANTICS

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ABSTRACT. A formulation of semantic theories for processes which does not rely on the notion of observables or convergence, is studied. The new construction is based solely on a reduction relation and equational reasoning, but can induce meaningful theories for processes, both in *weak* and *strong* settings. The resulting theories in many cases coincide with, and sometimes generalise, observation-based formulation of behavioural equivalence. The basic construction of reduction-based theories is studied, taking a simple name passing calculus (called ν-calculus) and its extensions as an example. Results concerning the application of our construction to other calculi are also briefly discussed.

1. INTRODUCTION

The aim of the present paper is to investigate a general formulation of process semantics which induces canonical congruence over agents based solely on a reduction relation and equational reasoning. Our starting point is the formulation of operational semantics for processes based on the reduction relation and structural rules introduced by Berry and Boudol [3], and later used by Milner [18]. By regarding structural rules as an embedded algebra for processes (cf. α-equality), we can view the reduction relation as representing the basic mechanism of computation in a given formalism. In the area of higher-order process calculi [20, 18, 19, 7], it was found that the reduction-based formulation enjoys much more simplicity than the transition-based one. It remains an open question, however, as to whether we can have a general framework for deriving meaningful process equivalences from the new construction which are as significant as well-known behavioural equivalences over agents e.g. bisimilarities [23, 16].

What we are going to show in the subsequent sections is that a general framework for deriving significant congruences over processes from the new construction in a mathematically elegant way, does exist. While work with an aim similar to ours has already appeared [21, 4], the equational framework in the present paper has several significant properties not found in those predecessors. First, the basic formal apparatus is an extension of the well-studied method found in both strict and lazy λ-theories [2, 1, 22], using the maximality condition among a certain family of congruences to derive canonical equality over agents. Specifically a fundamental element of the construction is *reduction-closure for equality*, which generalises β-equality (or convertibility) into the regime of concurrent processes. The basic idea underlying the convertibility is that, in harmony with the notion of mathematical functions which terms are supposed to denote, the meaning of a term does not change by reduction Such syntactic properties of reduction as the Church-Rosser property are also consistent with this idea. But in processes, the meaning of a term *does* change during computation. Thus the basic idea underlying the reduction closure is that the equality notion should be consistent with the state change induced by reduction. Second, one essential methodological appeal of the new construction is that we obtain those canonical (equational) theories *without committing ourselves to a specific notion of observation* — we even do not employ any kind of *convergence predicate* [21, 4], which is considered to be "minimised observability" in [21]. Rather, we use a semantic scheme analogous to the one used effectively in strict and lazy λ-calculi [2, 1, 22], where, in each setting, the identification of "meaningless" λ-terms in equational theories leads to a certain canonical equality which is more general than convertibility and, moreover, which coincides with a syntax-free model of the calculus. It turns out that the mechanism works well in the concurrency setting, inducing a notion of observability as a result of such identification. We note that, in such a semantic scheme, one stipulates what should be *equated* rather than what should be *differentiated* (the latter being the case when we positively stipulate observability conditions), hence no *a priori* semantic separation of agents is implied. Yet we shall show that the equations induced in this way often coincide with the behavioural equivalence produced by labeled transition relations, and sometimes capture additional equations which are behaviourally significant. Not only is the result theoretically interesting, but also the framework may be pragmatically useful when we try to define a semantics for a concurrent programming language where a definite notion of observation is not easily formulated, while a reduction relation can be naturally given, e.g. parallel languages with shared store or with complex interaction primitives [12, 24, 26]. We hope that the presented construction is helpful in such situations.

The present exposition develops the reduction-based semantic framework taking a small process calculus called ν-calculus [7, 4, 8, 10], an offspring of π-calculus [20, 18], together with its extensions, as the main object of the study. Simple as it is, this calculus enjoys a striking expressive power [4, 7, 10], and relates to certain formal systems which may be regarded as an analogue of combinatory logic in the concurrency setting [8, 9]. What is interesting in the present context is that the calculus can be given two distinct notions of observation, both of which seem equally reasonable, but which induce quite different transition relations and bisimilarities. When the reduction-based semantic framework is applied to the calculus, a canonical equality over terms is derived solely relying on a reduction relation and equational reasoning, whose equations capture behaviourally significant additional equations not found in bisimilarities. Reduction-based semantic construction is further investigated by applying the same semantic framework to some extensions of the ν-calculus, which include variants of the fragmentary π-calculus in [18]. The extensions incorporate additional operators on top of the basic calculus, and stronger observability (hence discernibility) is incrementally obtained in the corresponding canonical equalities as we add each operator. In the process we also restore two weak bisimilarities based on labeled transition relations in the purely reduction-based construction. It will also be shown how "strong" semantics (i.e. those which are concerned with the number of reduction steps) can be derived from the "weak" theories in a uniform way. The strong theories induced from the canonical weak theories correspond to strong bisimilarities from two transition relations. The results of the application of the reduction-based semantic framework to other formalisms are also summarised.

The remainder of the paper is organised as follows. *Section 2* presents the preliminaries to the subsequent study, including the basic definition of the ν-calculus, its equational theories, and its bisimilarities. *Section 3* constitutes the core part of the paper, culminating in the *observability theorem* (Theorem 3.19) and, as its consequence, the derivation of a certain canonical theory $\Im_\nu^*$ (Proposition 3.23). *Section 4* further explores the reduction-based equational construction. After showing that $\Im_\nu^*$ induces behaviourally interesting equations not included in the bisimilarities from two labeled transition relations (Proposition 4.3), we increment the basic calculus with two simple operators, the *match operator* and the *output synchronisation*, separately and in combination, and study the resulting canonical equalities. Three distinct observability theorems (Propositions 4.9, 4.13, 4.17) are obtained, and, as a consequence, the restoration results of two weak bisimilarities follow (Propositions 4.11, 4.18). Figure 1 in 4.19 summarises the results in the section. *Section 5* then studies the construction of strong theories based on the theories given in Sections 3 and 4, obtaining strong observability theorems (Propositions 5.5, 5.9) as well as the restoration of two strong bisimilarities (Propositions 5.7, 5.11). *Section 6* quickly summarises the results of applying the reduction-based construction to two other conspicuous calculi, CCS and λ-calculus. *Section 7* gives comparisons with the preceding work and presents some open issues.

Note. The paragraphs titled "Remark" may safely be skipped without losing consistency, while those titled "Discussion" form an integral part of the technical development.

2. ν-CALCULUS

Terms and Reduction

2.1. The ν-calculus, an offspring of the π-calculus [18, 20], is a succinct concurrency formalism using the notion of asynchronous name passing as the interaction primitive [7, 8, 10]. This simple primitive, coupled with the capability of replication of terms, gives the ν-calculus enough power to construct versatile interaction structures, just as a single operation called *β-reduction* in λ-calculus provides enough power for constructing any imaginable applicative behaviour. Below we introduce the basic notions of the calculus as far as needed for the present exposition.

2.2. ν-terms. Let $\mathbf{N}$ be a countable set of *names*, ranged over by $a, b, c, ..$, and $\mathbf{V}$ be a countable set of *name variables*, or simply *variables*, ranged over by $x, y, z, ..$, each disjoint from the other. The symbols $u, v, w, ..$ range over their union, the set of *identifiers*. The set of ν-terms $\mathbf{T}_\nu$, ranged over by $P, Q...$, is given by the following grammar.

$$P \ ::= \ \leftarrow uv \ \mid \ ux.P \ \mid \ P, Q \ \mid \ a \blacktriangleright P \ \mid \ !ux.P \ \mid \ \Lambda$$

Among terms, "$\leftarrow uv$" denotes a *message* to a target u carrying a value v. The construct can be regarded as a degenerated form of the output guard $\bar{u}v.P$ [20, 18], or as a reformulation of the notion of "argument" (in the sense of λ-calculus) in the name passing framework. Then "$ux.P$" denotes a *receptor* which receives a message and instantiates the value in its body. In $ux.P$, x binds free occurrences of x in P (like x in $\lambda x.M$). Here "ux" is called a *prefix* or an *input guard.* Another binding construct, "$a \blacktriangleright P$", is called a *scope restriction* where the initial a binds its free occurrences in P. "P, Q" is a concurrent composition of P and Q. $!ux.P$ is called a *lazy replicator* which replicates its body each time it interacts, where x is a binding occurrence. Finally "Λ" (to be read "nil") is a syntactic convention to denote that there is no agent.

2.3. Remark. A basic version of the ν-calculus uses another replicator, called the *simple replicator*, written $!_u P$. The construct is as powerful as the above replicator which comes from [20], or as the one given in [18] (see [9]). We use the present replicator since this version is often more convenient for codifying agents. The difference is inessential in the subsequent technical development.

2.4. Binding and substitution. We continue the introduction of the basic syntactic notions. First, the set of *free* (resp. *bound*) *names* in P is denoted by $\mathcal{FN}(P)$ (resp. $\mathcal{BN}(P)$). The set of *free* (resp. *bound*) *variables* in P is denoted by $\mathcal{FV}(P)$ (resp. $\mathcal{BV}(P)$). We also assume the usual notion of (multiple) substitution, written $\{\tilde{v}/\tilde{u}\}$, where $\tilde{u}$ is *either* a string of pairwise distinct variables, in which case $\tilde{v}$ should be a string of identifiers of the same length, *or* a string of pairwise distinct names, in which case $\tilde{v}$ should also be a string of names of the same length. Thus, within the present exposition, each substitution always maps variables to identifiers or else names to names, and no other cases are considered. $\equiv_\alpha$ denotes α-conversion regarding both bound names and bound variables. *Closed* and *open terms* are defined as usual (with respect to name variables). Thus $\leftarrow ab$ and $ax. \leftarrow xb$ are closed but $\leftarrow yc$ and $ax. \leftarrow cz$ are open. Then, for an open term P, a substitution σ which maps variables to names is a *closing substitution* if $\mathcal{FV}(P\sigma) = \emptyset$. This extends to a set of terms, where any term in the set is closed by the substitution.

Some conventions: $ab \blacktriangleright P$ denotes $a \blacktriangleright (b \blacktriangleright P)$. "," is the weakest in association, e.g. $(a \blacktriangleright P,\ Q) \overset{\text{def}}{=} ((a \blacktriangleright P),\ Q)$, and associates to the left. The outermost parentheses are often omitted. We write $\leftarrow c$ and $c.P$ to mean we do not care the value to be communicated. In spite of these conventions, we will freely use parenthesis to be explicit about syntax structure.

Now let the set of *contexts*, ranged over by C, C'..., be defined by:

$$C \ ::= \ [\ ] \ \mid \ a \blacktriangleright C \ \mid \ (C,\ P) \mid (P,\ C) \ \mid \ ux.C \ \mid \ !ux.C$$

Then an equivalence relation over $\mathbf{T}_\nu$ $\cong$ is a *congruence relation* if $P \cong Q$ implies $C[P] \cong C[Q]$ for any context C, and, moreover, $P \cong Q$ implies $P\{v/x\} \cong Q\{v/x\}$ for any v and x. Among the family of congruence relations, the following forms the core of the calculus.

2.5. Definition. (*structural congruence*) $\equiv$ is the smallest congruence relation over ν-terms generated by the following rules.

(i) $P \equiv Q$ if $P \equiv_\alpha Q$

(ii) $P, Q \ \equiv \ Q, P \quad (P, Q), R \ \equiv \ P, (Q, R) \quad P, \Lambda \ \equiv \ P$

(iii) $aa \blacktriangleright P \ \equiv \ a \blacktriangleright P \quad ab \blacktriangleright P \ \equiv \ ba \blacktriangleright P \quad a \blacktriangleright \Lambda \ \equiv \ \Lambda \quad a \blacktriangleright P, Q \ \equiv \ a \blacktriangleright (P, Q)$ if $a \notin \mathcal{FN}(Q)$.

The idea of building a process theory on the basis of equivalence classes of terms modulo such an equality originated with Berry and Boudol [3] and is essential for the definition of a reduction relation without a labeled transition system. The rules (iii) above come from Milner [18].

2.6. Reduction. The reduction relation provides the basic notion of computation in the formalism. The underlying idea is a quite primitive notion of interaction where a piece of information is simply consumed by an agent, which in turn generates a new agent after substitution. One step reduction relation over ν-terms is the smallest relation generated by:[1]

$$\begin{array}{ll}
(\mathrm{COM}) & ux.P,\ \leftarrow uv \ \rightarrow\ P\{v/x\}.\\
(\mathrm{REP}) & !ux.P,\ \leftarrow uv \ \rightarrow\ !ux.P,\ P\{v/x\}.\\
(\mathrm{PAR}) & P \ \rightarrow\ Q \ \Rightarrow\ P,R \ \rightarrow\ Q,R.\\
(\mathrm{RES}) & P \ \rightarrow\ Q \ \Rightarrow\ a\blacktriangleright P \ \rightarrow\ a\blacktriangleright Q.\\
(\mathrm{STR}) & P \equiv P' \quad P' \rightarrow Q' \quad Q \equiv Q' \ \Rightarrow\ P \rightarrow Q.
\end{array}$$

Then the multi-step reduction relation, $\twoheadrightarrow$, is defined by: $\twoheadrightarrow \overset{\text{def}}{=} \rightarrow^{*} \cup \equiv$.

We call terms of the form "$ux.P,\ \leftarrow uv$" or "$!ux.P,\ \leftarrow uv$" *redex*, and each term in one redex is called another's *redex pair*. Below we give some examples of reduction, along with several important expressions.

2.7. Example of reduction.

(i) Let $P \oplus Q \overset{\text{def}}{=} c\blacktriangleright(\leftarrow c,\ c.P,\ c.Q)$ with c fresh. Then:

$$P \oplus Q \rightarrow (P, c\blacktriangleright c.Q) \text{ and } P \oplus Q \rightarrow (Q, c\blacktriangleright c.P).$$

Note $c\blacktriangleright c.Q$ never reduces nor interacts.

(ii) Let $\mathsf{FW}(ab) \overset{\text{def}}{=} ay.\leftarrow by.$ Then:

$$\mathsf{FW}(ab), \leftarrow ac \ \rightarrow\ \leftarrow bc.$$

$\mathsf{FW}(ab)$ is called a *forwarder*, which forwards a message from port a to port b.

(iii) Let $\mathsf{EQ}(ab) \overset{\text{def}}{=} (!\mathsf{FW}(ab), !\mathsf{FW}(ba))$. Then:

$$\mathsf{EQ}(ab), \leftarrow ac \ \rightarrow\ \mathsf{EQ}(ab), \leftarrow bc \ \rightarrow\ \mathsf{EQ}(ab), \leftarrow ac \ \rightarrow \ldots$$

$\mathsf{EQ}(ab)$ is called an *equator*. Note that $\mathsf{EQ}(ab) \equiv \mathsf{EQ}(ba)$.

(iv) Let $\mathsf{I}(a) \overset{\text{def}}{=} !\mathsf{FW}(aa)$. Then:

$$\mathsf{I}(a), \leftarrow ac \ \rightarrow\ \mathsf{I}(a), \leftarrow ac \ \rightarrow\ \mathsf{I}(a), \leftarrow ac \ \rightarrow \ldots.$$

$\mathsf{I}(a)$ is called an *identity receptor*, which creates the same message even it interacts.

(v) Let $\mathsf{SW}(ae) \overset{\text{def}}{=} az.\leftarrow ze.$ Then:

$$\mathsf{SW}(ae),\ \leftarrow ab \ \rightarrow\ \leftarrow be.$$

$\mathsf{SW}(ab)$ is called a *switcher*, which sends back the value it holds to the name it receives.

(vi) Let $\Omega \overset{\text{def}}{=} o\blacktriangleright(\mathsf{I}(o),\ \leftarrow oo)$. Then:

$$\Omega \ \rightarrow\ \Omega \ \rightarrow\ \Omega \ \rightarrow\ \Omega \ \rightarrow\ \Omega \ \rightarrow \ldots.$$

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[1] If we use the simple replicator (cf. 2.3), the rule (REP) becomes: (REP) $!_u P,\ \leftarrow uv \ \rightarrow\ !_u P,\ P$.

Labeled Transitions and Bisimilarities

2.8. From this subsection to 2.14, we summarise the basic constructions and results concerning the behavioural equivalences of ν-calculus based on two labeled transition relations. The set of labels common to two labeled transition relations, ranged over by $l, l', ..$, is given by:

$$l \;\; = \;\; \tau \mid \;\downarrow ab \mid \;\uparrow ab \mid \;\uparrow a(b)$$

where "(b)" in "$\uparrow a(b)$" is the *bound* occurrence of the label. We write $\mathcal{BN}(l)$ and $\mathcal{FN}(l)$ for the sets of bound and free names in l. One important notion concerning labels is *relevance*: a label l is *relevant to P* if $\mathcal{BN}(l) \cap \mathcal{FN}(P) = \emptyset$. Now we give two sets of rules which induce transition relations over closed ν-terms, each of which represents a particular notion of observation.

2.9. Definition. (*synchronous transition relation*) The synchronous transition relation, denoted by $\xrightarrow{l}_s$, is the smallest relation over closed ν-terms inferred by the following rules.

IN$_s$: $\quad ax.P \xrightarrow{\downarrow ab}_s P\{b/x\}$

IN$'_s$: $\quad !ax.P \xrightarrow{\downarrow ab}_s \;!ax.P,\; P\{b/x\}$

OUT: $\quad \leftarrow ab \xrightarrow{\uparrow ab}_s \Lambda$

COM: $\quad ax.P,\; \leftarrow ab \xrightarrow{\tau}_s P\{b/x\}$

REP: $\quad !ax.P, \leftarrow ab \xrightarrow{\tau}_s \;!ax.P,\; P\{b/x\}$

PAR: $\quad P \xrightarrow{l}_s P' \;\;\Rightarrow\;\; R,\; P \xrightarrow{l}_s R,\; P' \qquad (\mathcal{BN}(l) \cap \mathcal{FN}(R) = \emptyset)$

RES: $\quad P \xrightarrow{l}_s P' \;\;\Rightarrow\;\; a \blacktriangleright P \xrightarrow{l}_s a \blacktriangleright P' \qquad (a \notin \mathcal{FN}(l) \cup \mathcal{BN}(l))$

OPEN: $\quad P \xrightarrow{\uparrow ab}_s P' \;\;\Rightarrow\;\; b \blacktriangleright P \xrightarrow{\uparrow a(b)}_s P' \qquad (a \neq b)$

STR: $\quad P' \equiv P,\; P \xrightarrow{l}_s Q,\; Q \equiv Q' \;\;\Rightarrow\;\; P' \xrightarrow{l}_s Q'$

Then $\overset{\hat{l}}{\Longrightarrow}_s$ stands for $\twoheadrightarrow$ if $l = \tau$, else for $\twoheadrightarrow \xrightarrow{l}_s \twoheadrightarrow$.

2.10. Definition. (*asynchronous transition relation*) The asynchronous transition relation, denoted by $\xrightarrow{l}_a$, is the smallest relation over closed ν-terms generated by the rules in Definition 2.9, using $\xrightarrow{l}_a$ instead of $\xrightarrow{l}_s$, deleting IN$_s$ and IN$'_s$ rules, and adding:

$$\text{IN}_a\text{:} \quad \Lambda \;\; \xrightarrow{\downarrow ab}_a \;\; \leftarrow ab$$

Then, as before, $\overset{\hat{l}}{\Longrightarrow}_a$ stands for $\twoheadrightarrow$ if $l = \tau$, else for $\twoheadrightarrow\xrightarrow{l}_a\twoheadrightarrow$.

2.11. Remark. The synchronous rules, specifically IN_s and IN'_s rules, are standard rules for process calculi, and regard the synchronisation to be essential in observation. In contrast, the IN_a rule concerns the asynchronous character of $\xrightarrow{l}_a$, signifying that an observer can send a message to an agent whenever as necessary. Note also the symmetry between IN_a and OUT rules we obtain in this way.

2.12. Bisimulations and bisimilarities. The following notions of bisimulation and bisimilarity are standard, except the relevance of labels is considered.

- (i) A *synchronous strong bisimulation* is any symmetric relation $\mathcal{R}$ over closed terms such that, if $P\ \mathcal{R}\ Q$, whenever $P \xrightarrow{l}_s P'$ with l relevant to Q, there exists Q' such that $Q \xrightarrow{l}_s Q'$ and $P'\ \mathcal{R}\ Q'$. Then an *asynchronous strong bisimulation* is similarly defined by replacing $\xrightarrow{l}_s$ with $\xrightarrow{l}_a$.
- (ii) A *synchronous weak bisimulation* is any symmetric relation $\mathcal{R}$ over closed terms such that, if $P\ \mathcal{R}\ Q$, whenever $P \xrightarrow{l}_s P'$ with l relevant to Q, there exists Q' such that $Q \overset{\hat{l}}{\Longrightarrow}_s Q'$ and $P'\ \mathcal{R}\ Q'$. Then an *asynchronous weak bisimulation* is similarly defined replacing $\xrightarrow{l}_s$ with $\xrightarrow{l}_a$ and $\overset{\hat{l}}{\Longrightarrow}_s$ with $\overset{\hat{l}}{\Longrightarrow}_a$.
- (iii) For each type of bisimulations, we can easily show, by the standard argument, that there exists the maximum bisimulation which is the union of all the bisimulations. The maximum bisimulation in each setting is called, with respective adjectives, *bisimilarity*. So we have synchronous strong/weak bisimilarities for closed terms (written $\sim^0_s$ / $\approx^0_s$), asynchronous strong/weak bisimilarities for closed terms (written $\sim^0_a$ / $\approx^0_a$). If P and Q are related by one of the bisimilarities, we say P and Q are *bisimilar*, again with an appropriate prefix.

The notion of bisimilarity naturally extends to the case of open terms. Given P and Q, one or both being open, if $P\sigma$ and $Q\sigma$ are bisimilar after any closing substitution w.r.t. one of the bisimilarities defined above, we say that they are bisimilar w.r.t. the bisimilarity. We extend the four bisimilarities in this way, and denote them by $\sim_s$ (from $\sim^0_s$), $\approx_s$ (from $\approx^0_s$), $\sim_a$ (from $\sim^0_a$), and $\approx_a$ (from $\approx^0_a$), respectively.

For these bisimilarities, we know:

2.13. Proposition. (bisimilarities)

- (i) $\sim_s$, $\approx_s$, $\sim_a$ *and* $\approx_a$ *are congruence relations.*
- (ii) $\sim_s \subsetneq \sim_a$ *and* $\approx_s \subsetneq \approx_a$. *Moreover* $\sim_a$ *and* $\approx_s$ *are incompatible.*

PROOF: See Appendix A. $\Box$

We note the following differentiating pairs: for the strong case, $(\mathsf{I}(a), \Omega) \not\sim_s \Omega$ but $(\mathsf{I}(a), \Omega) \sim_a \Omega$. For the weak case, $\mathsf{I}(a) \not\approx_s \Lambda$, but $\mathsf{I}(a) \approx_a \Lambda$. For incompatibility, we have $(\mathsf{I}(a), \Omega) \not\approx_s \Omega$. Note that the reduction of $\mathsf{I}(a)$ in Example 2.7 (iv) shows that $\mathsf{I}(a)$ does not give any effect to the outside, though in fact it engages in interaction.

2.14. Discussion. Thus different notions of observables give rise to different ways of equating terms, within a uniform framework of bisimilarity. We should note that, in so far as we rely on labeled transition relations or related notions, we *cannot* decide which of these two paradigms of observation is more "basic" to the calculus. While this relates to the essential merit of the notion of observables, i.e. applicability to varied purposes, a universal construction is necessary when, for example, we compare the expressiveness of two calculi. Note also that the notion of *convergence* as employed in [4, 21] cannot be the answer since we obtain different equivalences depending on whether we employ "asynchronous convergence" (where we only take output into account) or "synchronous convergence" (where we take both input and output into account). This is not what we can decide *within* the convergence-based semantic framework. These considerations motivate the formal framework in the subsequent sections.

ν-Theories

2.15. ν-theory. Equations over ν-terms are the central topic in the next three sections. While to directly deal with congruence relations among terms is possible, it is sometimes more convenient to treat equations as generated by a formal theory, e.g. when we want to be explicit about construction of a certain congruence.

A ν-theory, or simply a theory, is a formal theory, their formulae of the form $P = Q$, with at least the following axioms and rules.

$$\begin{array}{llll}
(1) & P = Q \quad \text{when } P \equiv Q & (6) & P = Q \;\Rightarrow\; c \blacktriangleright P = c \blacktriangleright Q \\
(2) & P = Q \;\Rightarrow\; Q = P & (7) & P = Q \;\Rightarrow\; ux.P = ux.Q \\
(3) & P = Q,\ Q = R \;\Rightarrow\; P = R & (8) & P = Q \;\Rightarrow\; !ux.P = !ux.Q \\
(4) & P = Q \;\Rightarrow\; P, R = Q, R & (9) & P = Q \;\Rightarrow\; P\{v/x\} = Q\{v/x\} \\
(5) & P = Q \;\Rightarrow\; R, P = R, Q & &
\end{array}$$

2.16. Remark (1). By Rule (1) of the above definition, all ν-theories include equations from structural equivalence as axioms. In one sense, this is because the multi-step reduction relation, which constitutes the core of the reduction-based equational construction from the next section, includes $\equiv$ as its subrelation (specifically, without Rule (1), the minimum theory, as defined in 2.18 (iv), would not be reduction-closed in the sense of 3.1). We may also understand the incorporation to be quite similar to the identification of α-convertible terms as in the λ-theories, cf. [2]. In each case, the basic algebra makes the definition of reduction, which constitutes the main dynamics of the formal system, much simpler than without it. This is related to the existence of graph-based representation which precisely corresponds to equivalence classes of terms modulo the equations, again in each case.

2.17. Remark (2). An essential point for introducing variables for names is shown in Rule (9) of 2.15, where we close the equality under substitution of name variables (cf. 2.12). Not only is this the standard notion in equational theories and thus allows us to use many familiar ideas developed already, but it has significant consequences algebraically. In π-calculi, cf. [20], the difference between names and variables is intentionally eliminated (cf. [6]), which *a posteriori* results in the requirement that behavioural congruences in general should be closed under name substitution. But the distinction of one from the other is the essential import of the notion of names, so the requirement should be regarded as a quite significant property of relations over processes, not one which process equalities in general possess. This would give another argument for our standard formulation which induces no such requirement as a consequence, allowing us to treat it separately from the general properties of congruences. Note also, as is well known, that the scope restriction operator "preserves" the original algebra over names, and thus is in harmony with the present formulation which uses name variables.

2.18. Notation.

(i) $\Im$, $\Im'$... range over ν-theories. The ν-theory with no additional axioms and rules is denoted by $\Im_{\equiv}$.

(ii) If $P = Q$ is provable in $\Im$ then we write $\Im \vdash P = Q$, or simply $P = Q$ when the concerned theory is obvious from the context. If $P = Q$ is not derivable from $\Im$, we write $\Im \not\vdash P = Q$.

(iii) Given a set of equations $\mathcal{E}$, $\mathcal{E} + \Im$ is the result of adding equations as axioms to $\Im$. $\mathcal{E}+$ denotes $\mathcal{E} + \Im_{\equiv}$. $\Im + \Im'$ is the result of adding the set of equations from two theories as axioms to the above rules.[2] We extend this to an arbitrary family of ν-theories, writing $\sum\{\Im_i\}_{i\in I}$, I being an index set.

(iv) The congruence relation induced by a theory $\Im$ is denoted by $|\Im|$. Given a family of ν-theories, the *maximum*, *minimum*, *maximal*, or *minimal* theories are those whose corresponding relations are the maximum, minimum, maximal, or minimal in that family. We say a ν-theory $\Im$ is a *subtheory* of another

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[2] Essentially the sum is the congruent closure of the union of congruences (or the lub in the lattice of congruences).

theory $\Im'$ if $|\Im| \subset |\Im'|$. If the inclusion is strict then the former is a *proper subtheory* of the latter.

(v) We say a theory is *consistent* if it does not equate all possible pairs of terms. A theory is *inconsistent* if it is not consistent.

2.19. Theories corresponding to bisimilarities. Given Definition 2.15, we can form theories corresponding to bisimilarities in 2.12, by adding equations from each bisimilarity to the minimum theory as axioms. We denote them by $\Im_{\sim s}$, $\Im_{\sim a}$, $\Im_{\approx s}$, and $\Im_{\approx a}$, respectively. Note we easily have $|\Im_{\sim s}| = \sim_s$, $|\Im_{\approx s}| = \approx_s$, $|\Im_{\sim a}| = \sim_a$, and $|\Im_{\approx a}| = \approx_a$, since each bisimilarity contains $\equiv$ as its subset [show $\equiv$ restricted to closed terms is a bisimulation], and since each is congruent by Proposition 2.13 (i).

2.20. Discussion. Barendregt [2] defines the notion of λ-*theories* as equational theories which are closed under the formation of λ-terms and which contain β-equality as an axiom scheme. Our formulation is quite similar to λ-theories, i.e. Rule (1) of 2.15 corresponds to λ's $\equiv_\alpha$ (cf. 2.1.12 of *op. cit.*), and Rules (4)–(8) correspond to the congruence rules of λ-theories. One significant difference is that, while the notion of λ-theory has β-equality as an axiom, incorporating the β-reduction into equality and closing the equality under substitution, Definition 2.15 does not possess such a rule. Note, however, that we *cannot* add the rule $P \rightarrow Q \;\Rightarrow\; P = Q$ in this setting, since such equality notion is too strong for concurrent processes, e.g. $P \oplus Q$ in Example 2.7 (i) reduces either to a term behaviourally equal (e.g. up to $\sim_s$) to P or to one equal to Q. Hence, the rule makes the theory inconsistent, at least semantically. This very example, however, motivates the closure property which we are going to study.

3. BASIC CONSTRUCTIONS

Reduction Closure Property

3.1. Reduction closure. The notion of *state* is essential in concurrency. Since a term would change its meaning during its reduction, "equality" in this setting means that two equated terms can go to an equated state again. That is, if two terms are ever equated and one is subsequently reduced to another term, then the other *should* be reducable to an equated term. In this way we require the notion of equation to be in harmony with the notion of reduction. The following formalises the idea.

Definition A ν-theory $\Im$ is *reduction-closed*, if, whenever $\Im \vdash P = Q$, $P \twoheadrightarrow P'$ implies, for some Q', $Q \twoheadrightarrow Q'$ and $\Im \vdash P' = Q'$.

Note that, if we had β-equality in the sense that $P \rightarrow P' \;\Rightarrow\; P = P'$, the closure property would be satisfied vacuously: thus the property generalises the β-equality into the regime of processes with a changeable state. The usage of $\twoheadrightarrow$ instead of $\rightarrow$ is essential in this respect; we have $II =_\beta I$ with $I \stackrel{\text{def}}{=} \lambda x.x$ in λ-calculus. In both, the underlying criterion is that the number of reduction steps should be abstracted in the semantic regime. From a pragmatic point of view, this is the very reason why semantic equality can provide a basis for program optimization, since, in this way, the concerns about equality are separated from the concerns on cost (cf. Section 5). For comparison of the above closure property with Milner-Sangiorgi's construction in [21, 19], see 7.2.

In the following, we often call reduction-closed ν-theories simply *reduction theories*. Two direct consequences of the above definition follow.

3.2. Proposition.

(i) $\Im$ *is reduction-closed if and only if, whenever* $\Im \vdash P = Q$, *for all* C, $C[P] \twoheadrightarrow P'$ *implies, for some* Q', $C[Q] \twoheadrightarrow Q'$ *with* $\Im \vdash P' = Q'$.

(ii) $\Im$ *is reduction-closed if and only if, whenever* $\Im \vdash P = Q$, $P \rightarrow P'$ *implies, for some* Q', $Q \twoheadrightarrow Q'$ *with* $\Im \vdash P' = Q'$.

PROOF: (i) is easy. For (ii), "only if" is immediate. For "if" direction, write $P \rightarrow^n P'$ when $P \equiv P'$ if $n = 0$, else $P \rightarrow^{n-1} \rightarrow P'$. Now $P \twoheadrightarrow P'$ iff, for some n, $P \rightarrow^n P'$. Then use induction on n. □

The clause (i) above makes one orientation of Definition 3.1 explicit, i.e. an aspect as a *test* on the internal consistency of an equational theory under arbitrary contexts, considering state changes of agents during reduction. The clause (ii) is often useful in proving the reduction-closure of a certain ν-theory.

We now state one essential property of reduction theories, after a lemma.

3.3. Lemma. (Chain Lemma) *Let* $\Im'$ *be* $\sum\{\Im_i\}_{i\in I}$, *a sum of a family of* ν*-theories. Then, if* $\Im' \vdash P = Q$, *we have a chain of equations:*

$$\Im_{j_0} \vdash P = R_0,\ ..,\ \Im_{j_k} \vdash R_{k-1} = R_k,\ ...,\ \Im_{j_n} \vdash R_{n-1} = Q$$

for some $j_i \in I$, $0 \leq i \leq n$.

PROOF: By induction on the rules of Definition 2.15 (cf. Lemma 16.1.6 (i) in [2]). □

3.4. Proposition. *Let* $\Im_i$ *be reduction-closed for all* $i \in I$. *Then* $\sum\{\Im_i\}_{i\in I}$ *is also reduction-closed.*

Proof: Let $\Im'$ be $\sum\{\Im_i\}_{i\in I}$. By the above lemma, if $\Im' \vdash P = Q$, we have a chain of equations:

$$\Im_{j_0} \vdash P = R_0,\ ..,\ \Im_{j_k} \vdash R_{k-1} = R_k,\ ..,\ \Im_{j_n} \vdash R_{n-1} = Q$$

with $j_k \in I$. Suppose $P \twoheadrightarrow P'$. Then for some R'_0 we have $R \twoheadrightarrow R'_0$ with $\Im_{j_0} \vdash P' = R'_0$. In this way we get:

$$\Im_{j_0} \vdash P' = R'_0,\ ..,\ \Im_{j_k} \vdash R'_{k-1} = R'_k,\ ..,\ \Im_{j_n} \vdash R'_{n-1} = Q'$$

with $R_k \twoheadrightarrow R'_k$ and $Q \twoheadrightarrow Q'$. By applying (3) in Definition 2.15 repeatedly we have $\Im' \vdash P' = Q'$ as required. □

The reasoning as given in the above proof is often used later, where we would just write "By Chain Lemma".

Note that the proposition does *not* imply the existence of the maximum consistent reduction-closed theory, since $\sum\{\Im_i\}_{i\in I}$ above can be inconsistent (note that an inconsistent theory is trivially reduction-closed). Indeed one unpleasant fact is that there is no maximum element in the family of consistent reduction-closed ν-theories, as will be proved in Proposition 3.9. A closely related fact is that equations in some reduction theories are pathological even if they are consistent, in spite of our intention to use the reduction-closure as the basic criteria for internal consistency of ν-theories. One example follows.

3.5. Proposition. *Define* $\Im_{\mathcal{G}} \stackrel{\text{def}}{=} \{(P,\, \mathcal{G}) = (Q,\, \mathcal{G}) \mid P, Q \in \mathbf{T}_\nu\}+$, *where* $\mathcal{G} \stackrel{\text{def}}{=} g \blacktriangleright \leftarrow gg$. *Then* $\Im_{\mathcal{G}}$ *is a consistent reduction-closed theory.*

To prove the proposition, we need a lemma, which is often useful. First, we introduce a notion of a class of formal theories which is defined just as in Definition 2.15 except, this time, theories do not necessarily include (7) and (8) of 2.15 as inference rules. Thus the set of equations of such a theory may not be always closed under the full context defined in 2.4. They are, however, closed under the set of *reduction contexts*, written C_r, C'_r, etc., which we define as:

$$C_r \ ::= \ [\,] \ \mid \ a \blacktriangleright C_r \ \mid \ (C_r,\, P) \mid (P,\, C_r)\ .$$

Let us call such a theory C_r*-theory*. The notion of reduction-closure is easily adapted to such a theory: a C_r-theory $\Im^-$ is reduction-closed if, whenever $\Im^- \vdash P = Q$ and $P \twoheadrightarrow P'$, there exists Q' such that $Q \twoheadrightarrow Q'$ with $\Im^- \vdash P' = Q'$.

3.6. Lemma. *Suppose* $\Im$ *is a (usual)* ν*-theory and* $\Im^-$ *is a* C_r*-theory, and, moreover, adding (7) and (8) of* 2.15 *to* $\Im^-$ *results in the same equations as* $\Im$*. Then if* $\Im^-$ *is reduction-closed,* $\Im$ *is also reduction-closed.*

PROOF: By the assumption, if $\mathcal{E}$ is the set of equations from $\Im^-$, we have $|\mathcal{E}+| = |\Im|$. Thus we safely assume that $\Im$ has exactly the equations from $\Im^-$ as its axioms and the rules of Definition 2.15 as its rules. Now we establish, for a pair P and Q such that $\Im \vdash P = Q$, the following property, using induction on the number of derivation steps.

- For any C_r and any substitution σ from variables to names, $C_r[P\sigma] \rightarrow P'$ implies, for some Q', we have $C_r[Q\sigma] \twoheadrightarrow Q'$ with $\Im \vdash P' = Q'$.
- Similarly $C_r[Q\sigma] \rightarrow Q'$ implies, for some P', we have $C_r[P\sigma] \twoheadrightarrow P'$ with $\Im \vdash Q' = P'$.

Then we use Proposition 3.2 (ii). Below (1),(2)... are the rules in Definition 2.15.

The base case is immediate. For the inductive step, if the last rule is (2), (4), (5), (6) or (9), the result is easy, while, if the rule is (3), we can use Chain Lemma. Thus the only interesting cases are (7) and (8). For (9), suppose $P = Q$ for which we have the above property. First we note that we can put, for some u' and σ', $(ux.P)\sigma \equiv_\alpha u'x.(P\sigma')$. We also note that, for some $\tilde{c}$ and R, we have $C_r[(ux.P)\sigma] \equiv \tilde{c}\blacktriangleright((ux.P)\sigma, R)$ with R being either a message, a receptor, or a replicator. Now suppose $C_r[(ux.P)\sigma] \rightarrow P'$, i.e. $\tilde{c}\blacktriangleright((ux.P)\sigma, R) \rightarrow P'$. Then we have one of the following.

(i) $R \rightarrow R'$ and $P' \equiv \tilde{c}\blacktriangleright((ux.P)\sigma, R')$,

(ii) $R \equiv (\leftarrow u'v, R')$ and $P' \equiv \tilde{c}\blacktriangleright((P\sigma')\{v/x\}, R')$.

In the case of (i), we have $C_r[(ux.Q)\sigma] \rightarrow \tilde{c}\blacktriangleright((ux.Q)\sigma, R')$. Moreover, $\tilde{c}\blacktriangleright((ux.P)\sigma, R') = \tilde{c}\blacktriangleright((ux.Q)\sigma, R')$ can be easily inferred in $\Im$ from $P = Q$. The case of (ii) is also easy since, if so, we have $C_r[(ux.Q)\sigma] \rightarrow \tilde{c}\blacktriangleright((Q\sigma')\{v/x\}, R')$, and then $\tilde{c}\blacktriangleright((P\sigma')\{v/x\}, R') = \tilde{c}\blacktriangleright((Q\sigma')\{v/x\}, R')$ is again easily inferred in $\Im$. The case when the last step is (8) is just similar. □

3.7. Remark. It is easy to see that the above proof can be generalised to an arbitrary property of the set of equations from a ν-theory (and, in another direction, to any partial contexts). Specifically, if we just want to know whether $|\Im|$ is a subset of a certain relation over ν-terms, say $\mathcal{R}$, we have the following schema, understanding the condition for $\Im^-$ as in Lemma 3.6: to prove $|\Im| \subset \mathcal{R}$, we show, first, that $|\Im^-| \subset \mathcal{R}$, second, that $\mathcal{R}$ is an equivalence relation (this corresponds to e.g. the use of Chain Lemma above), and, third, that $\mathcal{R}$ has the following properties: if $C_r[P\sigma]\ \mathcal{R}\ C_r[Q\sigma]$ for any σ from variables to identifiers and $C_r[\ ]$, then:

(i) $C_r[(ax.P)\sigma]\ \mathcal{R}\ C_r[(ax.Q)\sigma]$, and

(ii) $C_r[(!ax.P)\sigma]\ \mathcal{R}\ C_r[(!ax.Q)\sigma]$.

Since the reduction-contexts are often easier to reason about than the whole contexts, this gives us a useful proof technique, cf. Remark 3.25 and Appendix B.

3.8. Proof of Proposition 3.5. Consistency of $\Im_{\mathcal{G}}$ is easy by showing e.g. $\Im_{\mathcal{G}} \vdash P = Q$ implies either $P \equiv Q$ or $P \stackrel{\text{def}}{=} C_1[\mathcal{G}]$ and $Q \stackrel{\text{def}}{=} C_2[\mathcal{G}]$, by induction on the derivation steps. For the reduction-closure, we first prove $\Im_{\mathcal{G}}^-$, understood as in 3.6, is reduction-closed. Let us say P *contains an active garbage* when we can write $P \equiv C_r[\mathcal{G}]$. Then, by induction of derivation of equations, we easily know that $\Im_{\mathcal{G}}^- \vdash P = Q$ if and only if $P \equiv Q$ *or* P and Q both contain an active garbage. In the former case, $P \rightarrow P'$ implies $Q \rightarrow P'$, hence done. But in the latter case, if $P \twoheadrightarrow P'$ and P contains an active garbage, then P' also contains an active garbage by mechanically checking the rules of $\equiv$ and $\rightarrow$. So $\Im_{\mathcal{G}}^-$ is reduction-closed. Then we use Lemma 3.6. □

Since $\mathcal{G}$ is an utterly inactive term (for example we have $\mathcal{G} \sim_s \Lambda$), an agent $(P, \mathcal{G})$ behaves just like P in any context. Hence the fact that $\Im_{\mathcal{G}}$ equates $(P, \mathcal{G})$ and $(Q, \mathcal{G})$ for any P and Q, implies that the theory absurdly identify any two behaviour solely because both contain $\mathcal{G}$. There are infinitely many such theories. Moreover this proves:

3.9. Proposition. *There is no maximum element in the family of consistent reduction-closed ν-theories.*

PROOF: Remember $\Im_{\approx s}$. This is reduction-closed, hence if a maximum consistent reduction theory exists, $\Im_{\approx s} + \Im_{\mathcal{G}}$ is surely its (consistent) subtheory. But $\Im_{\approx s} + \Im_{\mathcal{G}} \vdash P = (P, \mathcal{G}) = (Q, \mathcal{G}) = Q$ for arbitrary $P, Q \in \mathbf{T}_\nu$, because $\mathcal{G} \approx_s \Lambda$ is easy by checking (the lack of) transition of these terms. So $\Im_{\approx s} + \Im_{\mathcal{G}}$ is inconsistent, a contradiction. □

3.10. Discussion. In the light of Proposition 3.9, as well as the existence of theories like $\Im_{\mathcal{G}}$, the reduction-closure property falls short of imposing a satisfactory constraint on ν-theories, both mathematically and intuitively (see Corollary 15.3.7 in [2] for a similar result regarding λ-theories). What further condition should we impose on ν-theories, which is both as semantically neutral as possible, and faithful to reduction-based construction? Here we take the idea inspired by the equational scheme of sensible λ-theories, where the identification of meaningless (unsolvable) terms *internally* filters out "unsound" equations, just because they would lead to inconsistency [2]. A similar equational scheme is found to be effective in the lazy λ-calculus (see [1, 22]). Meaninglessness in our context, however, cannot be given by such notions as *unsolvability* or *strong unsolvability*, which are specific to λ-calculi, but should be given in a way applicable to concurrency formalisms in general. The notion we introduce for this purpose is called *insensitivity*.

Insensitivity and Sound Theories

3.11. Insensitive terms. Insensitive terms are those which never interact with their enclosing contexts. The notion is intended to give a sufficient condition for operational insignificance of a term in the weak regime. Below let the set of *active names* of P, denoted by $\mathcal{AN}(P)$, be defined as: $\mathcal{AN}(\leftarrow ab) = \mathcal{AN}(ax.P) = \mathcal{AN}(!ax.P) = \{a\}$, $\mathcal{AN}(a \blacktriangleright P) = \mathcal{AN}(P) \backslash \{a\}$, $\mathcal{AN}(P, Q) = \mathcal{AN}(P) \cup \mathcal{AN}(Q)$, and $\mathcal{AN}(\Lambda) = \emptyset$.

Definition A closed term P is insensitive if, for all P' such that $P \twoheadrightarrow P'$, we have $\mathcal{AN}(P') = \emptyset$. An open term is insensitive if it is insensitive after any closing substitution. The set of insensitive terms is denoted by Ins_ν.

Paraphrasing, an agent is insensitive if its subterm never participates in reduction with the outside (i.e. never becomes a part of a redex whose pair lies outside of the agent). Some examples of (non-)insensitive terms follow.

3.12. Example. Λ and Ω are insensitive, but neither $\leftarrow ab$ nor $c \blacktriangleright (\leftarrow ce, cx. \leftarrow be)$ nor $\mathsf{I}(a)$ is insensitive.

3.13. Proposition.

(i) $P \in \mathsf{Ins}_\nu$ *and* $P \equiv P'$ *implies* $P' \in \mathsf{Ins}_\nu$.

(ii) $P \in \mathsf{Ins}_\nu$ *and* $P \twoheadrightarrow P'$ *implies* $P' \in \mathsf{Ins}_\nu$.

(iii) *For any* u, v, *if* $P \in \mathsf{Ins}_\nu$, *then* $P\{u/v\} \in \mathsf{Ins}_\nu$.

PROOF: (i) and (ii) (the latter implying the former) are immediate from the definition of Ins_ν. For (iii) we use the fact that $\mathcal{AN}(P) = \emptyset$ implies $\mathcal{AN}(P\{u/v\}) = \emptyset$ by the definition of $\mathcal{AN}(P)$, together with $P \twoheadrightarrow P' \;\Rightarrow\; P\{u/v\} \twoheadrightarrow P'\{u/v\}$. □

3.14. Generic reduction. To state one important property of insensitive terms, we need some preparation. First we remember the notion of *reduction contexts* from 3.5, ranged over by C_r, C'_r, ... Then a *generic reduction of a context* is defined as follows.

$$C_r \twoheadrightarrow C'_r \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall P.\ C_r[P] \twoheadrightarrow C'_r[P]$$

Intuitively speaking, when $C_r \twoheadrightarrow C'_r$ (which we read: C_r *generically reduces to* C'_r), the former context can reduce to the latter without any participation of the term in the hole. Using the idea, the next proposition formally tells us that insensitive terms actually "do nothing."

3.15. Proposition. (characterisation of insensitivity, cf. Proposition 14.3.24 in [2])
$\forall P \in \mathsf{Ins}_\nu \,.\, C_r[P] \twoheadrightarrow R \quad \Rightarrow \quad R \equiv C'_r[P'] \ \wedge\ C_r \twoheadrightarrow C'_r \ \wedge\ P \twoheadrightarrow P' \ \wedge\ P' \in \mathsf{Ins}_\nu \,.$

Proof: By induction on the number of reduction steps. □

This proposition says that whenever an insensitive term is placed in a context and whatever reduction takes place in the resulting configuration, we can collect no evidence that the term in a hole has participated in the change of the configuration at all. Note that the characterisation is given solely in terms of reduction relation and contextual closure, hence telling us the behavioural insignificance of insensitive terms independently of any (subjective) "observational commitment", as well as suggesting an alternative definition of insensitivity.

3.16. Sound theories. The fundamental mechanism we are going to introduce into our equational construction is the *identification of insensitive terms* in reduction-closed ν-theories. Proposition 3.15 provides us with a good reason for the identification. Thus we stipulate:

Definition A ν-theory is *sound* if it is consistent, reduction-closed, and, moreover, equates any two insensitive terms.

For example, $\Im_{\approx_s}$ and $\Im_{\approx_a}$ are sound. To prove this, reduction-closure is direct from the definition, while identification of insensitive terms in $\Im_{\approx_s}$ is established by showing that the symmetric closure of $\{\langle \Lambda,\ P\rangle \,|\, P \in \mathsf{Ins}_\nu\}$ with P closed, is a synchronous weak bisimulation [because: from P only $P \xrightarrow{\tau} P'$ is possible, but then P' is again in Ins_ν from Proposition 3.13 (ii)]. Thus $\Im_{\approx_s}$ is sound, then the soundness of $\approx_a$ easily follows from Proposition 2.13 (ii). In contrast, $\Im_{\sim_s}$ and $\Im_{\sim_a}$ are *not* sound, though they are obviously reduction-closed (cf. Section 5).

Generic Observables

3.17. Incompatible pairs. A pair of terms are *incompatible*, written $P \# Q$, if for any sound $\Im$, $\Im \not\vdash P = Q$ (cf. Definition 2.1.32 in [2]). Some such pairs become essential in deriving observability.

Lemma $\leftarrow c \# \Lambda$, $c.\Lambda \# \Lambda$, *and* $(\leftarrow c, c.\Lambda) \# \Lambda$.

Proof: To prove $\leftarrow c \# \Lambda$, first we show $\Im \vdash \leftarrow c = \Lambda \ \Rightarrow\ \Im \vdash \leftarrow c' = \Lambda$ for any c' with $\Im$ sound. Take a context: $C[\,] \overset{\text{def}}{=} c \blacktriangleright (c. \leftarrow c',\ [\,],\ \Lambda)$ with $c' \neq c$. Then $C[\Lambda] = \Lambda$ while $C[\leftarrow c] \rightarrow c \blacktriangleright (\leftarrow c',\ \Lambda) = (\leftarrow c',\ c \blacktriangleright \Lambda) = \leftarrow c'$. Since $\leftarrow c' \not\rightarrow$, we have $\leftarrow c' = \Lambda$ for arbitrary c', as required. Next we show $\leftarrow c \# \Lambda$ for arbitrary c. Suppose $\Im \vdash \leftarrow c = \Lambda$ in some sound theory. Let us be given any $P \in \mathbf{T}_\nu$. Then we can always find some c' such that $c' \notin \mathcal{FN}(P)$ (since $\mathcal{FN}(P)$ is always finite), and with this c' we construct a context $C[\,] \stackrel{\text{def}}{=} c' \blacktriangleright([\,],\ c'.P)$. By assumption and the result above, we have $\leftarrow c' = \Lambda$. Hence we should have $C[\Lambda] = C[\leftarrow c']$. But $C[\leftarrow c'] \rightarrow c' \blacktriangleright P = c' \blacktriangleright (P, \Lambda) = (P, c' \blacktriangleright \Lambda) = P$ while $C[\Lambda] = \Lambda$ hence $P = \Lambda$ for an arbitrary P, a contradiction. Hence $\leftarrow c \# \Lambda$.

For $c.\Lambda \# \Lambda$, we take $C[\,] \stackrel{\text{def}}{=} [\,], \leftarrow c$. Then $C[c.\Lambda] \rightarrow \Lambda$. But immediately $C[\Lambda] = \leftarrow c \not\rightarrow$. Hence $c.\Lambda \# \Lambda$ by the first incompatible pair.

To prove $(\leftarrow c, c.\Lambda) \# \Lambda$, take a context $C[\,] \stackrel{\text{def}}{=} c \blacktriangleright (c. \leftarrow e,\ [\,])$ with e fresh. Then $C[\leftarrow c,\ c.\Lambda] \rightarrow (\leftarrow e, c \blacktriangleright c.\Lambda) = \leftarrow e$, while $C[\Lambda] = \Lambda$, hence a contradiction. $\square$

Note that it is necessary, in the above proof, to have not only the reduction-closure property but also the identification of insensitive terms to derive incompatible pairs. This is in contrast to the fact that incompatible pairs in λ-calculus are established only under the assumption of β-equality, see Example 2.1.33 in [2]. Also note that the contexts used above are all reduction-contexts.

3.18. Generic observables. An essential fact about sound ν-theories is that *sound theories are, a posteriori, automatically provided with observables.* The induced observables are behaviourally significant, and, as studied in depth in 4.1 – 4.5, they well reflect important aspects of operations of the calculus. The obtained observables in turn bring about a certain canonical ν-theory as we shall see later.

The generic transition relation, denoted by $\overset{\uparrow a}{\leadsto}$, is the smallest relation over closed terms generated by the following rules, together with PAR and STR, replacing $\xrightarrow{l}_s$ with $\overset{\uparrow a}{\leadsto}$ in Definition 2.9.

$$\text{OUT:} \quad \leftarrow ab \overset{\uparrow a}{\leadsto} \Lambda$$

$$\text{RES:} \quad P \overset{\uparrow a}{\leadsto} P' \quad \Rightarrow \quad b \blacktriangleright P \overset{\uparrow a}{\leadsto} b \blacktriangleright P' \quad (a \neq b)$$

We define: $P \overset{\uparrow a}{\leadsto} \stackrel{\text{def}}{\Leftrightarrow} \exists P'.\, P \overset{\uparrow a}{\leadsto} P'$.

Note that the transition is only concerned with output messages, and, as such, only their targets, not values.

3.19. Theorem. (Main Theorem, Observability) *Let $\Im$ be a sound ν-theory and $\Im \vdash P = Q$, with P and Q closed. Then:*

$$P \overset{\uparrow a}{\leadsto} P' \quad \Rightarrow \quad Q \twoheadrightarrow \overset{\uparrow a}{\leadsto} \twoheadrightarrow Q' \quad \textit{for some } Q' \textit{ with } \Im \vdash P' = Q'.$$

Proof: First suppose $\Im \vdash P = Q$ for some sound theory $\Im$. We show that if $P \overset{\upharpoonright a}{\rightsquigarrow} P'$ then there should be some Q' such that $Q \twoheadrightarrow \overset{\upharpoonright a}{\rightsquigarrow} \twoheadrightarrow Q'$ with $\Im \vdash P' = Q'$, as far as $\Im$ is sound. Now take a context $C[\,] \overset{\mathrm{def}}{=} (a. \leftarrow c,\ c.\Lambda,\ [\,])$ with c fresh. Then easily $C[P] \rightarrow (\leftarrow c, c.\Lambda, P') \rightarrow P'$. Since $\Im$ is reduction-closed, we should have, for some Q'', $C[Q] \twoheadrightarrow Q''$ such that $\Im \vdash P' = Q''$. Now we show $Q \twoheadrightarrow \overset{\upharpoonright a}{\rightsquigarrow} \twoheadrightarrow Q''$. This is established by showing that Q'' cannot contain a term $c.\Lambda$ as its subterm.

But suppose $c.\Lambda$ is in Q'' as its subterm. Then let $\tilde{b}$ be such that $\{\tilde{b}\} \supset \mathcal{FN}(P) \cup \mathcal{FN}(Q)$ and $c \notin \{\tilde{b}\}$. Since $P' = Q''$ we have $\tilde{b} \blacktriangleright P' = \tilde{b} \blacktriangleright Q''$. But since Q'' contains $c.\Lambda$ it either contains $\leftarrow c$ or $a. \leftarrow c$ concurrently composed with $c.\Lambda$ [because: $c.\Lambda$ and $\leftarrow c$ as originally contained in $C[Q]$ can only interact with each other; moreover c cannot occur anywhere other than these terms during reduction, because c is never passed around]. Hence $\tilde{b} \blacktriangleright Q'' = c.\Lambda$ if $a. \leftarrow c$ never interacts or, with some Q''', $\tilde{b} \blacktriangleright Q'' \twoheadrightarrow \tilde{b} \blacktriangleright Q'''$ where $\tilde{b} \blacktriangleright Q''' = (c.\Lambda, \leftarrow c)$. But from Lemma 3.17 this is a contradiction, hence $c.\Lambda$ cannot be in Q''. But if $c.\Lambda$ is not in Q'' then there should be interaction between $a. \leftarrow c$ and a message in Q so that we have $Q \twoheadrightarrow \overset{\upharpoonright a}{\rightsquigarrow} \twoheadrightarrow Q''$, as required. □

3.20. Remark. Regarding Theorem 3.19 and its proof, we have the following observations.

(i) Note how an equation of two terms with discrepant behaviour results in inconsistency via the reduction-closure property and the identification of insensitive terms in the proof above. Thus soundness "generates sense" in equations. In essence, the acquisition of the observability relies on the existence of a certain context $C[\,]$, for each term R, which, if P and Q with discrepant behaviour are placed in the context, "amplifies" the difference so that $C[P]$ reduces to R while $C[Q]$ becomes insensitive (note reduction-closure is implicit here). This is also a formal mechanism at work in similar results below.

(ii) The above observability result, which neglects the input observable, is the *maximum* observability we can derive in general from the soundness condition, see 4.1 – 4.5.

(iii) To validate the existence of the maximum sound theory, as is done in Proposition 3.23 later, we have only to use the context $(a. \leftarrow c,\ [\,])$ with which we can obtain the weaker observability stated in Proposition 3.22 (i) below, hence also the maximum theory. The observability with closure in the above statement is, however, often useful for obtaining their correspondences with labelled transition relations not only in the ν-calculus but also in other calculi, as seen in the results presented in Sections 3, 4 and 6 (specifically see Proposition 6.7). Thus here and in later sections, we state (and prove) the observability results in their strongest possible form, including the closure property.

The Maximal Theory

3.21. Isolation. The essential technical consequence of the observability is the existence of *isolation sets.* Let $\mathbf{T}$ be a proper subset of $\mathbf{T}_\nu$. Then we say a theory $\Im$ *isolates* $\mathbf{T}$, or $\mathbf{T}$ is an *isolation set* in $\Im$, if $\Im \vdash P = Q$ and $P \in \mathbf{T}$ then $Q \in \mathbf{T}$. Using Chain Lemma we easily obtain:

Lemma *If $\Im_i$ isolates $\mathbf{T}$ for each $i \in I$, then $\sum\{\Im_i\}_{i \in I}$ also isolates $\mathbf{T}$.*

Note that if a theory isolates some non-empty set, then it is automatically consistent. The following properties are easily obtained.

3.22. Proposition.

(i) *Let $\Im$ be sound. Then it isolates the set $\{P \mid P \twoheadrightarrow \overset{\uparrow a}{\rightsquigarrow}\}$ for each $a \in \mathbf{N}$.*
(ii) *Let $\Im_i$ be sound for each $i \in I$. Then $\sum\{\Im_i\}_{i \in I}$ is also sound.*

Proof: As to (i), if $\Im \vdash P = Q$ and $P \twoheadrightarrow P' \overset{\uparrow a}{\rightsquigarrow}$, then $Q \twoheadrightarrow Q'$ and $\Im \vdash P' = Q'$ for some Q'. But by Theorem 3.19 $Q' \twoheadrightarrow \overset{\uparrow a}{\rightsquigarrow}$ hence $Q \twoheadrightarrow \overset{\uparrow a}{\rightsquigarrow}$. (ii) is immediate from (i) and Lemma 3.21. □

The existence of the maximum sound theory follows.

3.23. Proposition. (the maximum sound ν-theory) *Let $P \cong Q \overset{\mathrm{def}}{\Leftrightarrow} \Im \vdash P = Q$ for some sound $\Im$. We define: $\Im_\nu^* \overset{\mathrm{def}}{=} \{P = Q \mid P \cong Q\}+$. Then $\Im_\nu^*$ is the maximum sound theory, and is maximal among the whole family of reduction-closed theories.*

Proof: The theory is the sum of the whole family of sound ν-theories by definition. Then that it is sound follows from Proposition 3.22 (ii). But by construction there can be no consistent reduction-closed theory which is strictly greater than $\Im_\nu^*$ (in induced equations), hence the maximality. □

However, by Proposition 3.9, $\Im_\nu^*$ is *not* the maximum consistent reduction theory.

In the following we give a behavioural characterisation of the canonical theory, showing its operational significance. We use the *convergence predicate* in the asynchronous regime:

$$P \Downarrow \overset{\mathrm{def}}{\Leftrightarrow} \exists a.\ P \twoheadrightarrow \overset{\uparrow a}{\rightsquigarrow}\ .$$

3.24. Proposition. (behavioural characterisation of $\Im^*_\nu$) *We define relations* $\simeq_\kappa$ ($\kappa \in \mathbf{Ord}$, *the class of ordinals*), *by the following ordinal recursion*:

- $P \simeq_0 Q$ *if* $\forall C.\ C[P] \Downarrow \quad \Leftrightarrow \quad C[Q] \Downarrow$.
- $P \simeq_{\kappa+1} Q$ *if* $\forall C.$

$$\begin{array}{lcl} C[P] \twoheadrightarrow P' & \Rightarrow & \exists Q'.\ C[Q] \twoheadrightarrow Q' \simeq_\kappa P', \ \textit{and}, \\ C[Q] \twoheadrightarrow Q' & \Rightarrow & \exists P'.\ C[P] \twoheadrightarrow P' \simeq_\kappa Q' \end{array}$$

- *For limit* λ, $P \simeq_\lambda Q$ *if* $\forall \kappa < \lambda.\ P \simeq_\kappa Q$ (*i.e.* $\simeq_\lambda \ = \ \cap_{\kappa<\lambda} \simeq_\kappa$).

Then we have: $\cap_{\kappa \in \mathbf{Ord}} \simeq_\kappa \ = \ |\Im^*_\nu|$.

Proof: Let $\cap_{\kappa \in \mathbf{Ord}} \simeq_\kappa \overset{\text{def}}{=} \simeq$ and we define the function F, over subsets of $\simeq_0$, as follows. If $\mathcal{R} \subset \simeq_0$, then: $\langle P\,,\,Q\rangle \in F(\mathcal{R})$ if $\forall C.$

$$\begin{array}{lcl} C[P] \twoheadrightarrow P' & \Rightarrow & \exists Q'.\ C[Q] \twoheadrightarrow Q' \,\mathcal{R}\, P' \ \wedge \\ C[Q] \twoheadrightarrow Q' & \Rightarrow & \exists P'.\ C[P] \twoheadrightarrow P' \,\mathcal{R}\, Q'. \end{array}$$

Hence $\simeq_{\kappa+1} \ = \ F(\simeq_\kappa)$ with $\kappa \in \mathbf{Ord}$. Clearly F is monotonic, and, $\mathcal{R}$ is reduction closed congruence iff $\mathcal{R} \subseteq F(\mathcal{R})$ by definition. We first show $\simeq \supseteq |\Im^*_\nu|$ by transfinite induction. The base case $\simeq_0 \supseteq |\Im^*_\nu|$ is obvious by Propositions 3.2 (i) and 3.22 (i). For the inductive step, suppose $\simeq_\kappa \supseteq |\Im^*_\nu|$. Then:

$$\begin{array}{rcll} \simeq_{\kappa+1} = F(\simeq_\kappa) & \supseteq & F(|\Im^*_\nu|) & \text{(monotonicity of } F) \\ & \supseteq & |\Im^*_\nu| & (\Im^*_\nu \text{ is reduction closed}) \end{array}$$

Secondly, to show the other way round, note first that each $\simeq_\kappa$ is a congruence relation hence $\simeq$ is too. Let $\Im_\simeq$ is a theory corresponding to $\simeq$. Then $\Im_\simeq$ is a reduction-closed theory by the standard argument since for all R the set $\{R' \mid R \twoheadrightarrow R'\}$ is at most countable. So $\Im_\simeq$ is a sound theory, since the identification of insensitive terms is easy by the first inclusion. Hence $|\Im^*_\nu| \supseteq \simeq$, so $|\Im^*_\nu| = \simeq$, as required. □

The proposition tells us that asynchronous convergence, together with the reduction-closure property, fully characterises the canonical theory. One important remark is that the theory thus characterised is obtained *without presupposing any specific convergence notion*, not to say "$\Downarrow$" above. Apart from this, what we have obtained is the theory along the lines of applicative bisimulation in [1], but closed under all possible branching of multi-step reduction, thus incorporating the notion of changeable state.

3.25. Remark. We note, while we shall not use the result, that an analogue of Lemma 3.6 is also true for the soundness, i.e. if $\Im^-$ (understood as in 3.5) is sound, $\Im$ is also sound [since: easily one knows that the only non-trivial point is consistency. To check, let $P\mathcal{R}R \overset{\text{def}}{\Leftrightarrow} \forall a.\ (P \twoheadrightarrow\overset{\uparrow a}{\leadsto} \ \Leftrightarrow \ Q \twoheadrightarrow\overset{\uparrow a}{\leadsto})$ and use the proof schema of 3.7 e.g. since we can derive incompatible pairs as stated in Lemma 3.17 in this setting with precisely the same reasoning, we obtain the same observability result as in Theorem 3.19, hence $\Im^-$ isolates the set $\{P \mid P \twoheadrightarrow\overset{\uparrow a}{\rightsquigarrow}\}$ for each $a \in \mathbf{N}$, which shows $|\Im^-| \subset \mathcal{R}$, i.e. the base case is established. The inductive step is easy]. Note that this and Proposition 3.23 imply, if we construct the maximum theory in the setting of reduction-contexts, its equations coincide with those from $\Im^*_\nu$. The situation is somewhat analogous to what has been stated as a "context lemma" [15] in the setting of functional theories. The same is true in the subsequent development. To clarify a general condition under which such a property holds is an interesting open issue, cf. [11].

4. Operators and Observability

$\Im^*_\nu$ and Equators

4.1. In the following we try to explicate the equations by $\Im^*_\nu$ by showing its difference from $\Im_{\approx a}$, a theory based on the asynchronous weak bisimilarity. As we noted in 3.16, $\Im_{\approx s}$ and $\Im_{\approx a}$ are both sound, and, therefore, are subtheories of $\Im^*_\nu$. This fact gives us a tractable way of equating terms in $\Im^*_\nu$. Moreover $\Im_{\approx a}$ includes $\Im_{\approx s}$ in its equations (i.e. $|\Im^*_\nu| \supset \approx_a \supsetneq \approx_s$), and we easily obtain, by Theorem 3.19:

$$a.(b.c_1 \oplus b.c_2) \# a.b.c_1 \oplus a.b.c_2$$

with a, b, c distinct, showing how close sound theories are to the typical (weak) bisimulation (cf. Section 4 of Chapter 9 in [17]). So a natural question is whether there are any pairs of terms not equated by $\Im_{\approx a}$ but equated by $\Im^*_\nu$. Actually such pairs exist, and, interestingly, the additional equations capture one of the essential aspects of the name passing operation in the asynchronous regime. It is related with *equators* in Example 2.7 (iii): by a chain of equators, which continuously forward messages, we can *abstract locations* (i.e. two distinct names are semantically equated).

4.2. Definition. Let us write $P \overset{v\cdot w}{\leftrightarrow} Q$ for $P\{v/w\} \equiv Q\{v/w\}$, i.e. P and Q only differ in their (free) occurrences of v and w. We define[3]

$$\Im_{\mathsf{EQ}} \overset{\text{def}}{=} \{(P, \mathsf{EQ}(vw)) \;=\; (Q, \mathsf{EQ}(vw)) \mid v, w, P, Q \text{ arbitrary as far as } P \overset{v\cdot w}{\leftrightarrow} Q\} + .$$

Note that, if $(P, \mathsf{EQ}(ab)) \twoheadrightarrow\overset{\uparrow a}{\rightsquigarrow}$, then it is always the case $(P, \mathsf{EQ}(ab)) \twoheadrightarrow\overset{\uparrow b}{\rightsquigarrow}$. Thus, with respect to $\overset{\uparrow a}{\rightsquigarrow}$, equators operationally collapse two names. The following says that this property of equators is indeed semantically justifiable.

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[3] The same set of equations can be derived by adding the following rule to the minimum theory: $P\{y/x\} = Q\{y/x\} \;\Rightarrow\; (P, \mathsf{EQ}(xy)) = (Q, \mathsf{EQ}(xy))$. The present formulation is more convenient for the proof of Proposition 4.3.

4.3. Proposition. $\Im_{\mathsf{EQ}}$ *is a subtheory of* $\Im^*_\nu$.

PROOF: To prove $|\Im_{\mathsf{EQ}}| \subset |\Im^*_\nu|$, we show that the following three claims hold.

(a) $\Im_{\mathsf{EQ}}$ is reduction-closed.

(b) $\Im_{\mathsf{EQ}}$ isolates a set $\{P \mid P \twoheadrightarrow \overset{\uparrow a}{\rightsquigarrow}\}$ for each a.

(c) If $\Im$ is sound, then $\Im_{\mathsf{EQ}} + \Im$ is sound.

Note that (c) follows from (a) and (b) since, if so, $\Im_{\mathsf{EQ}} + \Im$ is reduction-closed and consistent, hence is immediately sound. Since a sound theory is always a subtheory of $\Im^*_\nu$, (c) implies that $\Im_{\mathsf{EQ}}$ is a subtheory of $\Im^*_\nu$. The establishment of (a) and (b) is left to Appendix B. $\square$

4.4. Corollary. $\Im_{\approx_a}$ *is a proper subtheory of* $\Im^*_\nu$.

PROOF: Because $\leftarrow ab \overset{b \cdot c}{\leftrightarrow} \leftarrow ac$, we have $\Im^*_\nu \vdash (\mathsf{EQ}(bc), \leftarrow ab) = (\mathsf{EQ}(bc), \leftarrow ac)$. But $(\mathsf{EQ}(bc), \leftarrow ab) \not\approx_a (\mathsf{EQ}(bc), \leftarrow ac)$ and $\Im_{\approx_a}$ is sound. $\square$

This explains why we only have observables which are without values to carry in Theorem 3.19. As to the exact difference between $\Im_{\approx_a}$ and $\Im^*_\nu$, we believe that the following holds.

4.5. Conjecture. $\Im_{\approx_a} + \{(\mathsf{EQ}(xy), \leftarrow zx) = (\mathsf{EQ}(xy), \leftarrow zy)\}$ *axiomatises* $\Im^*_\nu$.

The task of proving (or disproving) the statement is left as a future exercise.

Restoration of $\approx_a$

4.6. In the rest of the section, we investigate whether there is any systematic way of regaining $\approx_a$ and $\approx_s$ by the reduction-based formulation. The method leads to an interesting correspondence between the basic operators in a given calculus, on the one hand, and induced canonical equations, on the other hand. The basic idea is to increase the power of agents to discern behavioural difference by introducing additional syntactic operators. The study should be interesting as an exploration of the semantic significance of each operator in the name passing framework. A close correspondence between these operators and the discernibility of terms in the induced canonical theories is revealed, and summarised graphically at the end of the section. Two constructs from [20], *the match operator* and *the output synchronisation*, each separately and then in combination, are considered. We start with the former.

4.7. Definition. (*match operator* [20]) The match operator has the form "$[u = v]$", used as "$[u = v]P$", with a structural rule:

$$[u = u]P \equiv P \ .$$

Note that the operator directly compares two names, so that it might well annihilate the power of the equators. We call the system extended with the new syntactic construct and the structural rule, $\nu_=$. The new set of terms is denoted by $\mathbf{T}_{\nu=}$, which is a superset of $\mathbf{T}_\nu$. We use the same definition for the reduction relation, written again as $\longrightarrow$ and $\twoheadrightarrow$. For transition relation, we only consider $\xrightarrow{l}_a$ defined with the same rules over the new set of terms. The asynchronous weak bisimilarity in this setting is written $\approx_a^=$. For the bisimilarity we have:

4.8. Proposition.

(i) $\approx_a^= \ \cap \ (\mathbf{T}_\nu \times \mathbf{T}_\nu) \ = \ \approx_a$.

(ii) $\approx_a^=$ *is a congruence relation.*

PROOF: (i) is easy since $\approx_a^= \ \cap \ (\mathbf{T}_\nu \times \mathbf{T}_\nu)$ is, when restricted to closed terms, a bisimulation in $\mathbf{T}_\nu$, while $\approx_a$ is, again when restricted to closed terms, a bisimulation in $\mathbf{T}_{\nu=}$. (ii) is easily proved, constructing an appropriate closure in each case (cf. Appendix A). The proviso concerning bisimilarities over open terms as given at the end of 2.12 is essential to closure under e.g. the match operator, cf. [20]. □

For equational theories, which we call $\nu_=$-theories, we add:

$$P = Q \ \ \Rightarrow \ \ [u = v]P = [u = v]Q \ .$$

Then, using the incompatible pairs similar to those in Lemma 3.17, we can prove common isolation sets for the sound theories. This guarantees the existence of the maximum element, which is denoted by $\Im^*_{\nu=}$. Regarding this, an interesting observability result is known.

4.9. Proposition. (observability with the match operator)

(i) *Let $\Im$ be a sound $\nu_=$-theory and $\Im \vdash P = Q$ with both terms closed. Then, if $\mathcal{BN}(l) = \emptyset$,*

$$P \xrightarrow{l}_a P' \quad \Rightarrow \quad Q \overset{\hat{l}}{\Longrightarrow}_a Q' \quad \textit{for some } Q' \textit{ with } \Im \vdash P' = Q'.$$

(ii) *Suppose $\Im^*_{\nu=} \vdash P = Q$. Then, with l relevant to Q, we have:*

$$P \xrightarrow{l}_a P' \quad \Rightarrow \quad Q \overset{\hat{l}}{\Longrightarrow}_a Q' \quad \textit{for some } Q' \textit{ with } P' \approx_a^= Q'.$$

PROOF: (i) The case of τ is trivial by reduction closure. For the free input, suppose $P \xrightarrow{\downharpoonleft ab}_a (P, \leftarrow ab)$. Then $Q \xrightarrow{\downharpoonleft ab}_a (Q, \leftarrow ab)$, hence $\Im \vdash (P, \leftarrow ab) = (Q, \leftarrow ab)$ by congruence of $\Im$. For the free output, suppose $\Im \vdash P = Q$ and $P \xrightarrow{\upharpoonleft ab}_a P'$. Immediately $P \equiv (\leftarrow ab,\ P')$. Take a context $C[\,] \overset{\text{def}}{=} ([\,],\ ax.([x = b] \leftarrow c),\ c.\Lambda)$ with c fresh then we have $C[P] \rightarrow^2 P'$. This implies, by reduction closure property, for some Q', $C[Q] \twoheadrightarrow Q'$ with $\Im \vdash P' = Q'$. Now we show $Q \overset{\upharpoonleft ab}{\Longrightarrow}_a Q'$. But if $c.\Lambda$ still occurs in Q', we know, by the reasoning similar to the proof for Theorem 3.19 restricting free names except c, that we get either: $\leftarrow c = \Lambda$ or $(\leftarrow c, c.\Lambda) = \Lambda$ (note if $a \neq b$, then $[a = b]P = \Lambda$ by the definition of insensitive terms), a contradiction. Thus Q' does not contain $c.\Lambda$ which shows that there is a reduction, for some Q'',

$$C[Q] \twoheadrightarrow (Q'', \leftarrow ab, ax.([x = b] \leftarrow c), c.\Lambda) \rightarrow (Q'', \leftarrow c, c.\Lambda) \twoheadrightarrow Q'$$

but this shows $Q \overset{\upharpoonleft ab}{\Longrightarrow}_a Q'$ with $\Im \vdash P' = Q'$, concluding this case.

(ii) Using (i), we show that there exists an asynchronous weak bisimulation $\mathcal{S}$ such that $|\Im^*_{\nu=}| \subset \mathcal{S}$. See Appendix C for details. □

The proposition suggests that the strict semantic difference which the operator induces in the calculus, as already suggested by Boudol [4]. In our context, the operator provides the *capability* to discriminate two locations (or names), which results in the *inability* to abstract locations by equators.

4.10. Proposition. $\approx^=_a \; = \; |\Im^*_{\nu=}|$.

PROOF: It is easy to show $\approx^=_a \; \subseteq \; |\Im^*_{\nu=}|$ since Proposition 4.8 (ii) and the bisimulation property imply that $\approx^=_a$ corresponds to a reduction-closed $\nu_=$-theory with the common isolation sets with $\Im^*_{\nu=}$. By Proposition 4.9 (ii) the converse implication is obvious. □

The first restoration result follows.

4.11. Proposition. (restoration of $\approx_a$) *Define a ν-theory:*

$$\Im^*_{\nu=|\mathbf{T}_\nu} \overset{\text{def}}{=} \{P = Q \mid \Im^*_{\nu=} \vdash P = Q \;\wedge\; P, Q \in \mathbf{T}_\nu\}+$$

i.e. the restriction of $\Im^*_{\nu=}$ to the original ν-terms. Then $|\Im^*_{\nu=|\mathbf{T}_\nu}| = \approx_a$.

PROOF: Note that $|\Im^*_{\nu=|\mathbf{T}_\nu}| = |\Im^*_{\nu=}| \cap (\mathbf{T}_\nu \times \mathbf{T}_\nu)$ since the latter is obviously congruent in $\mathbf{T}_\nu$. Then the result is immediate from Propositions 4.8 (i) and 4.10. □

Restoration of $\approx_s$

4.12. A system with the output synchronisation: π-calculus. We next enrich the ν-calculus with the operator for output synchronisation, resulting in a system which is essentially the same as the fragmentary π-calculus in [18].[4] The syntax of terms is given by:

$$P \quad ::= \quad \bar{u}v.P \;\mid\; ux.P \;\mid\; !ux.P \;\mid\; P, Q \;\mid\; a \blacktriangleright P \;\mid\; \Lambda$$

Such notions as free/bound variables/names, substitution, open/closed terms are defined as in 2.4. The structural congruence $\equiv$ is defined with the same rules as in 2.5. To get the reduction relation, we replace (COM) and (REP) in Definition 2.6 with the following rules.

$$\begin{array}{rl} \text{(COM)} & ux.P,\, \overline{u}v.Q \;\rightarrow\; P\{v/x\}, Q \\ \text{(REP)} & !ux.P,\, \overline{u}v.Q \;\rightarrow !ux.P,\; P\{v/x\},\; Q \end{array}$$

We call the system *π-calculus* for simplicity. Note that, by regarding "$\bar{u}v.\Lambda$" as "$\leftarrow uv$", $\mathbf{T}_\nu$ is a proper subset of π-terms. Moreover one easily sees that both the structural congruence and the reduction relation of the system, when projected onto $\mathbf{T}_\nu$, coincide with those of the ν-calculus.

For equational theories,

$$P = Q \;\;\Rightarrow\;\; \bar{u}v.P = \bar{u}v.Q$$

is added, and all definitions in 2.18 are carried over to the new setting. We call theories defined in this way, *π-theories*. With the notion of active names reformulated by using $\mathcal{AN}(\bar{a}b.P) = \{a\}$ instead of $\mathcal{AN}(\leftarrow ab) = \{a\}$, we have the set of insensitive terms in π-calculus as in 3.11 and call it Ins_π. Consistent reduction-closed π-theories which identify insensitive terms are called *sound*.

Now a labeled transition relation $\xrightarrow{l}_\pi$, defined as the smallest relation over closed π-terms inferred by the rules in Definition 2.9, with $\xrightarrow{l}_s$ read as $\xrightarrow{l}_\pi$ except COM, REP and OUT being replaced by:

$$\begin{array}{rl} \text{COM:} & ax.P,\, \overline{a}b.Q \;\xrightarrow{\tau}_\pi\; P\{b/x\},\; Q \\[1ex] \text{REP:} & !ax.P,\, \overline{a}b.Q \;\xrightarrow{\tau}_\pi !ax.P,\; P\{b/x\},\; Q \\[1ex] \text{OUT:} & \bar{a}b.P \;\xrightarrow{\upharpoonleft ab}_\pi\; P \end{array}$$

Then $\sim_\pi$ and $\approx_\pi$ are the strong and weak bisimilarities based on the transition relation, respectively, defined as in 2.12. They are easily congruence relations over $\mathbf{T}_\pi$, and the corresponding π-theories are denoted by $\Im_{\sim\pi}$ and $\Im_{\approx\pi}$, respectively.

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[4] The difference in structural rules and replicators between the system here and the system in [18] is not significant regarding the subsequent results.

We also define another, weaker transition relation: $P \overset{\upharpoonleft a}{\rightsquigarrow} P'$ with P and P' closed π-terms, defined as in 3.18 replacing OUT with $\bar{a}b.P \overset{\upharpoonleft a}{\rightsquigarrow} P$, and also $P \overset{\downharpoonleft a}{\rightsquigarrow} P'$ if, for some b, $P \xrightarrow{\downharpoonleft ab}_{\pi} P'$.[5]

4.13. Proposition. *Let $\Im$ be a sound π-theory and $\Im \vdash P = Q$ with P and Q closed. Then, with $\overset{h}{\rightsquigarrow}$ denoting $\overset{\upharpoonleft a}{\rightsquigarrow}$ or $\overset{\downharpoonleft a}{\rightsquigarrow}$, if $P \overset{h}{\rightsquigarrow} P'$ we have, for some Q', $Q \twoheadrightarrow \overset{h}{\rightsquigarrow} \twoheadrightarrow Q'$ with $\Im \vdash P' = Q'$.*

Proof: The case $\overset{\upharpoonleft a}{\rightsquigarrow}$ is done by the same reasoning as in Theorem 3.19. For $\overset{\downharpoonleft a}{\rightsquigarrow}$, use the context $C[\,] \overset{\text{def}}{=} (\bar{a}.\bar{c}.\Lambda,\ c.\Lambda,\ [\,])$ with c fresh. $\square$

This tells us, via the corresponding isolation sets, that there exists the maximum element among sound π-theories in this setting. We call the theory $\Im^*_\pi$. To understand the equations from the theory, we again use the equators.

4.14. Proposition. *Let $\mathsf{EQ}_\pi(vw) \overset{\text{def}}{=} (!vx.\bar{w}x.\Lambda,\ !wx.\bar{v}x.\Lambda)$. Then, with $\Im_{\mathsf{EQ}_\pi}$ defined as $\{(P, \mathsf{EQ}_\pi(uv)) = (Q, \mathsf{EQ}_\pi(uv)) \mid P \overset{u \cdot v}{\leftrightarrow} Q\}+$, $\Im_{\mathsf{EQ}_\pi}$ is a subtheory of $\Im^*_\pi$.*

Proof: Note that $(\mathsf{EQ}_\pi(ac), \bar{a}b.P) \ \rightarrow^2\ (\mathsf{EQ}_\pi(ac), \bar{a}b.\Lambda, P)$. This reduction allows us to apply the reasoning in the proof of Proposition 4.3 to the present case, regarding $\bar{a}b.\Lambda$ as $\leftarrow ab$. $\square$

The use of the proof schema of Proposition 4.3 becomes possible because equators in the π-calculus, by their very existence, *transform* the synchronous communication into the asynchronous one, as shown by the reduction in the proof above (in fact, we even have: $\Im^*_\pi \vdash (\mathsf{EQ}_\pi(ac), \bar{a}b.P) \ =\ (\mathsf{EQ}_\pi(ac), \bar{a}b.\Lambda, P)$). Then they function as in the asynchronous regime, i.e. *equate* names.

4.15. Proposition.

(i) *$\Im_{\approx\pi}$ is a proper subtheory of $\Im^*_\pi$.*

(ii) *Let a ν-theory $\Im^*_{\pi|\mathbf{T}_\nu}$ be given by $\{P = Q \mid \Im^*_\pi \vdash P = Q \ \wedge\ P, Q \in \mathbf{T}_\nu\}+$. Then it is sound.*

(iii) *$\approx_s \subsetneq |\Im^*_{\pi|\mathbf{T}_\nu}| \subsetneq |\Im^*_\nu|$, while $\approx_a$ and $|\Im^*_{\pi|\mathbf{T}_\nu}|$ are incompatible.*

Proof:

(i) Easily $\Im_{\approx\pi}$ is sound. Then take a pair of $(\mathsf{EQ}_\pi(ab), \bar{e}a.\Lambda)$ and $(\mathsf{EQ}_\pi(ab), \bar{e}b.\Lambda)$.

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[5] The definition of $\overset{\downharpoonleft a}{\rightsquigarrow}$ from $ax.P \overset{\downharpoonleft a}{\rightsquigarrow} P\{b/x\}$ is difficult due to its interaction with RES rule.

(ii) Because any sound π-theory restricted to $\mathbf{T}_\nu$ is a sound ν-theory [since: let $\Im$ be a sound π-theory; then $|\Im|$ restricted to $\mathbf{T}_\nu$ is easily congruent and, because the reduction in $\mathbf{T}_\pi$ always takes the image of a ν-term to such an image, the theory is reduction-closed; but Ins_π restricted to $\mathbf{T}_\nu$ coincides with Ins_ν].

(iii) For inclusions, use $\approx_\pi \cap (\mathbf{T}_\nu \times \mathbf{T}_\nu) \ = \approx_s$, together with Proposition 4.14 for the first one; (ii) above for the second, together with $\Im^*_{\pi|\mathbf{T}_\nu} \not\vdash \mathsf{l}(a) = \Lambda$. For strictness, use the example listed in the proof of (i) above for the first, the pair of $\mathsf{l}(a)$ and Λ for the second. These two pairs together prove the incompatibility. □

4.16. π-calculus with the match operator. Finally, on top of π-calculus, we add the match operator again, which eventually leads us to the restoration of $\approx_s$ in the reduction-based framework. Hereafter "$\pi_=$" stands for the extension of the fragmentary π-calculus with the match operator. The set of terms are obtained by adding $[u = v]P$ to the grammar given in 4.12, with the structural rule $[u = u]P \equiv P$ added, just as in 4.7. Then the definition of the reduction relation is given just as in π-calculus, which we write $\to$ and $\twoheadrightarrow$. For transition relation, we only consider $\xrightarrow{l}_\pi$ defined with the same rules as in 4.12 over the new set of terms. The weak bisimilarity is denoted by $\approx_s^=$.

The equational theories are defined in a similar way, together with the notion of soundness. Observability in the line of e.g. Proposition 4.13 is easily obtained in the same way. This gives us the maximum sound theory in this system, which we call $\Im^*_{\pi=}$.

4.17. Proposition. *Suppose $\Im^*_{\pi=} \vdash P = Q$. Then, with l relevant to Q, we have:*

$$P \xrightarrow{l}_\pi P' \quad \Rightarrow \quad Q \overset{\hat{l}}{\Longrightarrow}_\pi Q' \quad \textit{for some } Q' \textit{ with } P' \approx_s^= Q'.$$

Proof: The proof is essentially the same as that of Proposition 4.9. Suppose $\Im$ is a sound $\pi_=$-theory and $\Im \vdash P = Q$. First we show that if $\mathcal{BN}(l) = \emptyset$ then $P \xrightarrow{l}_\pi P' \quad \Rightarrow \quad Q \overset{\hat{l}}{\Longrightarrow}_\pi Q'$ for some Q' with $\Im \vdash P' = Q'$. The only difference exists in the case of input in Proposition 4.9 (i). Suppose $P \xrightarrow{\downarrow ab}_\pi P'$ with $\Im \vdash P = Q$, and let us define $C[\ ] \overset{\text{def}}{=} ([\ ],\ \bar{a}b.\bar{c}.\Lambda,\ c.\Lambda)$ with c fresh. Then we have $C[P] \to (P',\ \bar{c}.\Lambda,\ c.\Lambda) \to P'$, and for some Q'', $C[Q] \twoheadrightarrow Q''$ with $\Im \vdash P' = Q''$. But if not $Q \overset{\downarrow ab}{\Longrightarrow}_\pi$, then we have $\tilde{f} \blacktriangleright P' = \Lambda = c.\Lambda = \tilde{f} \blacktriangleright Q''$ with $\{\tilde{f}\} \supset \mathcal{FN}(P) \cup \mathcal{FN}(Q)$ and $c \notin \{\tilde{f}\}$, a contradiction. The closure property is also satisfied with $(c.\Lambda, \bar{c}.\Lambda) \# \Lambda$.

Then we show that there exists a synchronous weak bisimulation $\mathcal{S}$ such that $|\Im^*_{\pi=}| \subset \mathcal{S}$. This part of the proof proceeds by checking each transition, using the exactly same contexts as those in Proposition 4.9 (ii) except in the case of the input transition, as explained in Appendix D. □

4.18. Proposition.

(i) $|\Im^*_{\pi=}| = \approx_s^=$.

(ii) (restoration of $\approx_s$) *Define a* ν*-theory* $\Im^*_{\pi=|\mathbf{T}_\nu} \stackrel{\text{def}}{=} \{P = Q \mid \Im^*_{\pi=} \vdash P = Q \wedge P, Q \in \mathbf{T}_\nu\}+$. *Then* $|\Im^*_{\pi=|\mathbf{T}_\nu}| = \approx_s$.

PROOF: Just as in the proofs of Propositions 4.10 and 4.11. □

4.19. Summary. By our development up to this point, we know that the lattice of enrichment of operators results in the lattice of observables, hence of canonical equalities. The following graphic depicts the situation. Below the observables are listed on the left of the name of the system, and the operators added to the base calculus are on the right. Dotted lines show (upward) inclusions in canonical equations, projected onto $\mathbf{T}_\nu$.

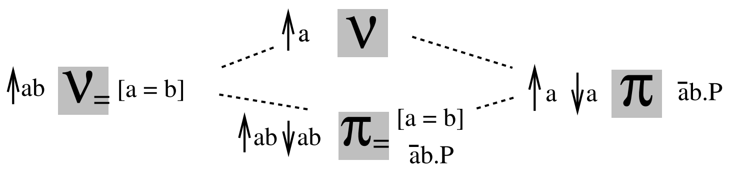

FIGURE 1. Observability and Operators.

Notice how incompatibility in operators between systems results in incompatibility between their observability (hence between their equations). When we have both the output synchronisation and the match operator, the equation becomes the most strict. The essential fact is that the picture has been obtained by a single scheme of equational construction: deeper study on operators and their semantic significance in process calculi is called for.

5. STRONG THEORIES

Basic Construction

5.1. Reduction theories are essentially conceived in the "weak" semantic framework, since the number of reduction steps, less the termination, is irrelevant. Notably we have $\Omega = \Lambda$ in any sound ν-theory. The following construction now tells us that we can easily derive strong theories out of the weak ones. In fact a canonical strong theory corresponds to each reduction theory, and, as will be seen, one such theory (derived from $\Im_\nu^*$) corresponds to the asynchronous strong bisimilarity introduced in 2.12. Thus, the framework moves from weak semantics to strong semantics, which may be contrary to the usual practice [16, 17, 21]: the idea here is to first try to capture the *meaning* of each term in the abstract setting, then refine the induced equations by considering the exact steps with which a term need to reach possible semantic points. *Cost* (i.e. the number of reduction steps) is measured after *sense* (i.e. abstract weak equality) is made, so to speak.

The basic definition of strong theories follows. Remember that a theory is a *subtheory* of another theory if its equations are included in the latter's.

5.2. Definition. (*strong theory*) Given a reduction theory $\Im$, we say its subtheory $\Im'$ is a *strong reduction theory with base* $\Im$, if, whenever $\Im' \vdash P = Q$, $P \rightarrow P'$ implies, for some Q', $Q \rightarrow Q'$ with $\Im' \vdash P' = Q'$.

Some easy yet useful facts:

5.3. Proposition.

(i) *Let $\Im'$ be a subtheory of $\Im$. Then $\Im'$ is a strong reduction theory with base $\Im$ if and only if, whenever $\Im' \vdash P = Q$, $P \rightarrow^n P'$ implies, for some Q', $Q \rightarrow^n Q'$ with $\Im' \vdash P' = Q'$, for all $n \in \omega$.*

(ii) *Suppose $\Im'$ is a strong theory with base $\Im$. If $\Im$ is consistent, then $\Im'$ is also consistent. If $\Im$ is moreover sound, then $P \# Q$ implies $\Im' \not\vdash P = Q$.*

(iii) *Let $\Im_1$ and $\Im_2$ be reduction-closed theories and $|\Im_1| \subset |\Im_2|$. Then if $\Im_1'$ is a strong theory with base $\Im_1$, there is always a strong theory $\Im_2'$ with base $\Im_2$ such that $|\Im'_1| \subset |\Im'_2|$.*

PROOF: Immediate. □

Note that (i) shows that a strong theory is derived from a base theory by "counting reduction steps". The following is important to get a (relatively) canonical strong theory from a base theory.

5.4. Proposition.

(i) *Let* $\Im_i$ *be a strong theory with base* $\Im$ *for all* $i \in I$*. Then* $\sum\{\Im_i\}_{i\in I}$ *is also a strong theory with* $\Im$ *as its base.*

(ii) *Given a reduction theory* $\Im$*, there is always the maximum theory among the strong theories with base* $\Im$*.*

(iii) *Let* $\Im$ *and* $\Im'$ *be reduction theories, and* $\Im_s$ *and* $\Im'_s$ *be their corresponding maximum strong theories. Then* $|\Im| \subseteq |\Im'|$ *implies* $|\Im_s| \subseteq |\Im'_s|$*.*

PROOF: (i) is easy by Chain Lemma. (ii) follows from (i). (iii) comes from Proposition 5.3 (iii). □

Restoration of $\sim_a$

5.5. Strong observability. Let $\Im^*_{\nu:s}$ denote the maximum strong theory corresponding to $\Im^*_\nu$. The essential result in this section follows. It tells us that, by counting reduction steps on the basis of the equations from $\Im^*_\nu$, we obtain the observability of values.

Proposition (observability in strong theories)

(i) *Let* $\Im$ *be a strong theory whose base is a sound theory and* $\Im \vdash P = Q$*. Then, if* $\mathcal{BN}(l) = \emptyset$*, we have:*

$$P \xrightarrow{l}_a P' \quad \Rightarrow \quad Q \xrightarrow{l}_a Q' \quad \textit{for some } Q' \textit{ with } \Im \vdash P' = Q'.$$

(ii) *Suppose* $\Im^*_{\nu:s} \vdash P = Q$*. Then, if* l *is relevant to* Q*, we have:*

$$P \xrightarrow{l}_a P' \quad \Rightarrow \quad Q \xrightarrow{l}_a Q' \quad \textit{for some } Q' \textit{ with } P' \sim_a Q'.$$

PROOF: (i) We prove the result reasoning by the number of reduction steps. The cases for τ and input are trivial. For the free output, suppose $\Im \vdash P = Q$ and $P \xrightarrow{\uparrow ab}_a P'$. Take a context $C[\,] \overset{\text{def}}{=} ([\,],\ ax.\leftarrow xc,\ by.(\leftarrow d, \leftarrow y),\ c.\Lambda,\ d.\Lambda)$ with c, d fresh. Then:

$$C[P] \ \rightarrow\ (P', \leftarrow bc,\ by.(\leftarrow d, \leftarrow y),\ c.\Lambda,\ d.\Lambda) \ \rightarrow^3\ P'$$

Thus there should be some Q' s.t. $C[Q] \rightarrow^4 Q'$ with $\Im \vdash P' = Q'$. Take a context $C'[\,] \overset{\text{def}}{=} \tilde{f} \blacktriangleright [\,]$ with $\{\tilde{f}\} \supset \mathcal{FN}(P) \cup \mathcal{FN}(Q)$ and $c, d \notin \{\tilde{f}\}$. Then $d.\Lambda$ and $c.\Lambda$ cannot be in Q' since, if so, e.g. $\Im^*_\nu \vdash d \blacktriangleright C'[Q'] = c.\Lambda$, while $\Im^*_\nu \vdash d \blacktriangleright C'[P'] = \Lambda$ but by Proposition 5.3 (ii) (the second part) and Lemma 3.17 this is absurd. Thus $\leftarrow d$ and $\leftarrow c$ should come out, so at the first step, there should be interaction in one of the following forms:

$$(1)\ C[Q] \ \rightarrow\ (Q',\ ax.\!\leftarrow\! xc,\ \leftarrow\! d, \leftarrow\! g,\ c.\Lambda,\ d.\Lambda) \ \text{ with } Q \xrightarrow{\upharpoonright bg}_a Q' \text{ for some } g,$$
$$(2)\ C[Q] \ \rightarrow\ (Q',\ \leftarrow\! g'c,\ by.(\leftarrow\! d, \leftarrow\! y),\ c.\Lambda,\ d.\Lambda) \text{ with } Q \xrightarrow{\upharpoonright ag'}_a Q' \text{ for some } g'.$$

But to eliminate $d.\Lambda$ and $c.\Lambda$ by three steps, (1) is impossible and $\leftarrow bc$ should be present at the stage (2). This means $g' = b$ hence $\leftarrow ab$ is present at the first step. Therefore $C[Q] \rightarrow^4 Q'$ where $\Im \vdash Q' = P'$ and $Q \xrightarrow{\upharpoonright ab}_a Q'$, as required.

(ii) As is done in the proof of e.g. Proposition 4.9 (ii), we show that there exists some asynchronous strong bisimulation $\mathcal{S}$ such that $\mathcal{S} \supseteq |\Im^*_{\nu:s}|$. The reasoning proceeds by checking each transition, using (i) above in an essential way. See Appendix E. □

5.6. Proposition. (restoration of $\sim_a$) $|\Im^*_{\nu:s}| = \sim_a$.

Proof: We first note that the theory $\Im_{\sim a}$ is a subtheory of $\Im^*_{\nu:s}$ [since: first we have $|\Im^*_\nu| \supset \approx_a \supset \sim_a$, hence $\Im_{\sim a}$ is a strong theory with base $\Im^*_\nu$, but $\Im^*_{\nu:s}$ is the maximum as such]. Then Proposition 5.5 (ii) tells us that $|\Im^*_{\nu:s}| \subset \sim_a$. □

This also gives us the following result for the maximum strong theory whose base is the asynchronous weak bisimilarity.

5.7. Proposition. *Let* $\Im_{\approx a:s}$ *be the maximum strong theory with base* $\Im_{\approx a}$*. Then* $|\Im_{\approx a:s}| = \sim_a$.

Proof: We know $|\Im^*_{\nu:s}| \supset |\Im_{\approx a:s}|$ by Proposition 5.4 (iii) together with $|\Im^*_\nu| \supset \approx_a$. But $|\Im_{\approx a:s}| \supseteq \sim_a$, since $\Im_{\sim a}$ is a strong theory with base $\Im_{\approx a}$ by $\sim_a \subset \approx_a$, hence the result is immediate from Proposition 5.6. □

Propositions 5.6 and 5.7 show that the difference between $\approx_a$ and $\Im^*_\nu$ depends on the use of additional reduction steps (e.g. equators): if these should be counted, the additional equations disappear. Hence the restoration of values in messages without the match operator in $\Im^*_{\nu:s}$, leading to the coincidence of the "strong core" in the two different weak theories.

Restoration of $\sim_s$

5.8. In light of the development reaching Propositions 5.6 and 5.7, we may expect that the maximum strong theories corresponding to $\Im^*_\pi$ and $\Im^*_{\pi=}$ induce the same set of equations, when projected onto $\mathbf{T}_\pi$ in the latter case, which also coincide with the equations from $\sim_\pi$. We verify this in the following. As will be seen later, the result can automatically be transferred to the case of $\Im^*_{\pi|\mathbf{T}_\nu}$ (the restriction of $\Im^*_\pi$ to $\mathbf{T}_\nu$) and $\Im_{\approx s}$.

We first construct a strong theory with base $\Im^*_\pi$ (in a way similar to strong theories for ν-calculus), denoted by $\Im^*_{\pi:s}$. Remember that $\sim_\pi$ is the strong bisimilarity over the π-terms derived from $\xrightarrow{l}_\pi$, cf. 4.12.

5.9. Proposition.

(i) *Suppose* $\Im^*_{\pi:s} \vdash P = Q$. *Then, if* l *is relevant to* Q, *we have:*

$$P \xrightarrow{l}_\pi P' \quad \Rightarrow \quad Q \xrightarrow{l}_\pi Q' \quad \textit{for some } Q' \textit{ with } P' \sim_\pi Q'.$$

(ii) *Let* $\Im_{\approx\pi:s}$ *be the maximum strong with base* $\Im_{\approx\pi}$. *Then:* $|\Im^*_{\pi:s}| = |\Im_{\approx\pi:s}| = \sim_\pi$.

Proof: Regarding (i), we first show the same property as in the proof of Proposition 5.5 (i). The difference in reasoning only lies in the input transition, which is easily verified using the same context as is found in the proof of Proposition 4.17, i.e. $C[\;] \overset{\text{def}}{=} ([\;],\ \bar{a}b.\bar{c}.\Lambda,\ c.\Lambda)$ with c fresh. Next the synchronous observability is derived from exactly the same contexts as in the proof of Proposition 5.5 (ii) (given in Appendix E) except in the case of the input transition, for which we use the context given in Appendix D (6). Then (ii) is obtained just as in Propositions 5.6 and 5.7, respectively. □

Let us say a π-theory $\Im'$ is an *expansion* of a ν-theory $\Im$ when it is constructed with axioms and rules of the minimum π-theory plus exactly the set of equations from $\Im$ (projected onto $\mathbf{T}_\pi$) as the additional axioms. Using these notions, the following is needed before the restoration of $\sim_s$.

5.10. Proposition. (expansion) *Let* $\Im'$ *be the expansion of a strong* ν*-theory* $\Im$. *Then* $\Im'$ *is a strong* π*-theory. Moreover if the same* $\Im$ *has a base which is the restriction of a* π*-theory, the latter becomes a base of* $\Im'$.

Proof: The verification of the first part uses the schemata similar to that of Lemma 3.6, where we take the set of contexts which do not use the output prefix instead. Then we can mechanically prove, if a theory based on such a partial context is strong in terms of its equations, then its expansion to the whole context is also a strong theory. The second part is easy using the fact that if a ν-theory $\Im_1$ is the restriction of a π-theory $\Im_2$ to ν-terms, then the expansion of $\Im_1$ has the same equations as $\Im_2$, as well as the fact that if a ν-theory is greater than another ν-theory, then the expansion of the former is greater than that of the latter. □

5.11. Proposition. (restoration of $\sim_s$) *The congruence relations from the following* ν*-theories coincide with* $\sim_s$.

(i) $\Im^*_{\pi:s}|_{\mathbf{T}_\nu} \overset{\text{def}}{=} \{P = Q \mid \Im^*_{\pi:s} \vdash P = Q \ \wedge\ P, Q \in \mathbf{T}_\nu\}+$,

(ii) $\Im_{\approx\pi:s|\mathbf{T}_\nu} \overset{\mathrm{def}}{=} \{P = Q \mid \Im_{\approx\pi:s} \vdash P = Q \ \wedge \ P, Q \in \mathbf{T}_\nu\}+$,

(iii) $\Im^*_{\pi|\mathbf{T}_\nu:s}$, *which is the maximum strong theory with base* $\Im^*_{\pi|\mathbf{T}_\nu}$, *and*

(iv) $\Im_{\approx s:s}$, *which is the maximum strong theory with base* $\Im_{\approx s}$.

PROOF: For (i), use $\sim_\pi \cap(\mathbf{T}_\nu \times \mathbf{T}_\nu) = \sim_s$, together with Proposition 5.9 (ii). Regarding (ii), by the reasoning as in the proof of Proposition 5.7, it follows that $|\Im^*_{\pi:s}| = |\Im_{\approx\pi:s}|$, then use (i). For (iii), note that, if we restrict any strong theory with base $\Im^*_{\pi:s}$ to $\mathbf{T}_\nu$, obviously it is a strong theory with base $\Im^*_{\pi|\mathbf{T}_\nu}$. At the same time, any strong theory with base $\Im^*_{\pi|\mathbf{T}_\nu}$ can be expanded to a strong theory with base $\Im^*_\pi$ by Proposition 5.10, hence is also a subtheory of $\Im^*_{\pi|\mathbf{T}_\nu}$. Therefore $|\Im^*_{\pi:s|\mathbf{T}_\nu}|$ and $|\Im^*_{\pi|\mathbf{T}_\nu:s}|$ coincide, thus from (i) we are done. (iv) is similar. □

6. RESULTS ON OTHER CALCULI

CCS

6.1. CCS [16, 17] is the classical process calculus using dyadic synchronisation as the basic communication primitive. The system has been studied by Milner and Sangiorgi [21] regarding a semantic formulation based on reduction and convergence. Below we summarise the results of the application of our reduction-based construction to CCS, where we easily restore the two well-known bisimilarities by considering theories closed under the reduction contexts. For the purpose of comparison and also for simplicity, we use essentially the same syntax as [21] (without relabeling). Since the formulation of the reduction relation of CCS without labeled transition relation does not seem to appear in the literature, we give basic definitions of the calculus in this line.

6.2. Terms. With $\mathcal{A}$ being the set of names, ranged over by $a, b, c, ..$, $\mathbf{L}$ is the set of labels defined as $\{a, \overline{a} \mid a \in \mathcal{A}\}$, ranged over by $l, l' \ldots$. The set of terms, $\mathbf{T}_{ccs}$, ranged over by $P, Q, \ldots$, is given by:[6]

$$P \ ::= \mathbf{0} \ \mid \ l.P \ \mid \ P\backslash L \ \mid \ P|Q \ \mid \ P + Q \ \mid \ A$$

where $L \subset \mathcal{A}$ and names in L bind the free occurrences of the same names and their co-names of P in $P\backslash L$. We use $\mathcal{FN}(P)$ for the set of free names (the *syntactic sort* of Definition 2.2 in [17]). α-equality is denoted by $\equiv_\alpha$ with respect to the binding by "$\backslash$". The symbol A stands for a process constant to which we assume an associated defining equation $A \overset{\mathbf{def}}{=} P$. $(((P\backslash L_1)\backslash L_2)\backslash L_3)$ is written $P\backslash L_1 L_2 L_3$.

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[6] We omit the τ action since "$\tau.P$" can be defined as "$(a.P|\bar{a}.\mathbf{0})\backslash\{a\}$" where a is fresh, as noted in [19]. The restriction to the finite summation is not substantial.

6.3. Definition. (*structural congruence*) $\equiv$ is the smallest congruence relation over CCS-terms generated by the following rules.

(i) $P \equiv Q \quad \text{if } P \equiv_\alpha Q$.

(ii) $P \mid Q \equiv Q \mid P, \quad (P \mid Q) \mid R \equiv P \mid (Q \mid R), \quad P \mid \mathbf{0} \equiv P$.

(iii) $P \backslash L \mid Q \equiv (P \mid Q) \backslash L \ \text{ if } \{L \cup \overline{L}\} \cap \mathcal{FN}(Q) = \emptyset$.

(iv) $A \equiv P \ \text{ if } A \overset{\mathbf{def}}{=} P$.

(v) $P + Q \equiv Q + P, \quad (P + Q) + R \equiv P + (Q + R)$.

(vi) $P \backslash L_1 L_2 \equiv P \backslash L_1 \cup L_2$.

6.4. Definition. (*reduction*) One step reduction relation over CCS-terms is the smallest relation generated by:

$$\begin{array}{ll}
(\mathrm{COM}) & (l.P + R) \mid (\bar{l}.P' + Q) \ \to \ P \mid P' \\
(\mathrm{SUM}) & P \ \to \ P' \ \Rightarrow \ P + R \ \to \ P' \\
(\mathrm{PAR}) & P \ \to \ P' \ \Rightarrow \ P \mid R \ \to \ P' \mid R \\
(\mathrm{RES}) & P \ \to \ P' \ \Rightarrow \ P \backslash L \ \to \ P' \backslash L \\
(\mathrm{STR}) & P' {\equiv} P, \ P \to Q, \ Q {\equiv} Q' \ \Rightarrow \ P' \to Q'
\end{array}$$

Then $\twoheadrightarrow$ is defined as $\to^{*} \cup \equiv$.

We also adopt the *labeled transition relation* over $\mathbf{T}_{ccs}$ from [21], denoted $\xrightarrow{\alpha}$, where α is either a label or τ, and the corresponding weak transition relation, $\xRightarrow{\hat{\alpha}}$. Note $\to \ = \ (\xrightarrow{\tau}\equiv)$, and $\twoheadrightarrow \ = \ (\xRightarrow{\varepsilon}\equiv)$. Accordingly the strong bisimilarity, $\sim$, and the weak bisimilarity, $\approx$, are defined in the standard way.

6.5. Reduction theories in CCS. We next define reduction theories for CCS, taking $\equiv$ as base equations. One essential issue in constructing reduction theories for CCS is that the summation is problematic in weak congruent theories in general, making significant equations such as $\mathbf{0} = (\bar{a}.\mathbf{0} | a.\mathbf{0}) \backslash \{a\}$ impossible. The issue can be solved by relativising the construction by a subset of contexts, i.e. we construct theories except rules for the summation and the prefix, just as is done in [21].

A ccs-theory, or simply a theory, is a formal theory, their formulae of the form $P = Q$, with at least the following axioms and rules.

(1) $P = Q \quad \text{when } P \equiv Q$

(2) $P = Q \ \Rightarrow \ Q = P$

(3) $P = Q, \ Q = R \ \Rightarrow \ P = R$

(4) $P = Q \ \Rightarrow \ P \mid R = Q \mid R$

(5) $P = Q \ \Rightarrow \ R \mid P = R \mid Q$

(6) $P = Q \ \Rightarrow \ P \backslash L = Q \backslash L$

The reduction-closure property is defined as in 3.1. Let us define the set of *active names* of P, denoted by $\mathcal{AN}(P)$ as: $\mathcal{AN}(a.P) = \{a\}$, $\mathcal{AN}(\overline{a}.P) = \{\overline{a}\}$, $\mathcal{AN}(P\backslash L) = \mathcal{AN}(P) - \{L \cup \overline{L}\}$, $\mathcal{AN}(A) \stackrel{\rm def}{=} \mathcal{AN}(P)$ if $A \stackrel{\rm def}{=} P$, $\mathcal{AN}(P+Q) = \mathcal{AN}(P \mid Q) = \mathcal{AN}(P) \cup \mathcal{AN}(Q)$, and $\mathcal{AN}(\mathbf{0}) = \emptyset$. Then insensitive terms are given as Definition 3.11. As in 3.16, we easily know that if P is insensitive, then $P \approx \mathbf{0}$. Under these notions, we obtain the sound theories, denoted by $\Im, \Im'$..., like in the ν-calculus.

6.6. Lemma. (incompatible pairs) *$l.\mathbf{0} \# \mathbf{0}$ and $(l.\mathbf{0} \mid l'.\mathbf{0}) \# \mathbf{0}$.*

PROOF: For $l.\mathbf{0} \# \mathbf{0}$, if $l = a$, take a context $C[\ ] \stackrel{\rm def}{=} (\overline{a}.c.\mathbf{0} \mid [\ ])\backslash\{a\}$ for any c. Then it is easy by the similar reasoning as Lemma 3.17 with the context $C'[\ ] \stackrel{\rm def}{=} (\overline{c}.P \mid [\ ])$ with $\overline{c} \notin \mathcal{FN}(P)$. If $l = \overline{a}$, we change $\overline{a}$ and c to a and $\overline{c}$ respectively in the above context. For the proof of $(l.\mathbf{0}|l'.\mathbf{0}) \# \mathbf{0}$, if $l' = \overline{l}$, this corresponds to $(\leftarrow c, c.\Lambda) \# \Lambda$ of ν-calculus, so it is obvious. Next let $l' \neq \overline{l}$. We take a context $C[\ ] \stackrel{\rm def}{=} (\overline{l}.\overline{c}.\mathbf{0} \mid \overline{l'}.\mathbf{0} \mid [\ ])\backslash\{l, l'\}$ with c fresh. Then $C[(l'.\mathbf{0} \mid l.\mathbf{0})] \twoheadrightarrow (\overline{c}.\mathbf{0} \mid \mathbf{0} \mid \mathbf{0})\backslash\{l, l'\} \equiv \overline{c}.\mathbf{0} \mid (\mathbf{0} \mid \mathbf{0})\backslash\{l, l'\} = \overline{c}.\mathbf{0}$. But $C[\mathbf{0}]$ is insensitive, hence a contradiction. □

The main result in this subsection follows. Since the communication operation of CCS is a simple synchronisation (without the value passing), the generic observability corresponding to e.g. Theorem 3.19 in ν-calculus directly lets us obtain a weak bisimulation.

6.7. Proposition. (observability in CCS) *Suppose $\Im$ is a sound theory and $\Im \vdash P = Q$. Then we have, whenever $P \xrightarrow{\alpha} P'$, there is some Q' such that $Q \stackrel{\hat{\alpha}}{\Longrightarrow} Q'$ and $\Im \vdash P' = Q'$. In other words, $|\Im|$ is a weak bisimulation.*

PROOF: We check each transition system. The case $\alpha = \tau$ is obvious from $\xrightarrow{\tau} = \longrightarrow\equiv$ and $\equiv \subset |\Im|$. For the case $\alpha = a$, we use the context $C[\ ] \stackrel{\rm def}{=} (\overline{a}.\overline{c}.\mathbf{0} \mid c.\mathbf{0} \mid [\ ])$ with c fresh, then apply the same reasoning as Theorem 3.19. The case $\alpha = \bar{a}$ is just symmetric to the previous case. Hence $|\Im|$ is a weak bisimulation. □

6.8. Corollary. *The maximum sound theory exists. Call it $\Im^*_{ccs}$. Then $|\Im^*_{ccs}|$ coincides with $\approx$.*

PROOF: By Chain Lemma (or using obvious isolation sets), the maximum sound theory, $\Im^*_{ccs}$, exists. Immediately $|\Im^*_{ccs}|$ is a bisimulation, hence $|\Im^*_{ccs}| \subset \approx$. Also $\approx$ is closed under reduction context (by Proposition 5.11 of [17]), equates insensitive terms, and is reduction-closed by definition. Therefore we have $|\Im^*_{ccs}| \supset \approx$, as required. □

Finally we show the reasoning to establish coincidence of $\sim$ and the maximum strong theory derived from $|\Im^*_{ccs}|$.

6.9. Proposition. (characterisation of $\Im^*_{ccs:s}$) *Write* $\Im^*_{ccs:s}$ *for the maximum strong theory with base* $\Im^*_{ccs}$. *Then* $|\Im^*_{ccs:s}|$ *coincides with* $\sim$.

Proof: We first show $\Im^*_{ccs:s} \subset \sim$. Assume $\Im$ be a strong theory whose base is sound and $\Im \vdash P = Q$. We first check $P \xrightarrow{\alpha} P' \;\Rightarrow\; Q \xrightarrow{\alpha} Q'$ for some Q' with $\Im \vdash P' = Q'$. For the case $\alpha = \tau$, it is obvious from the definition of the strong reduction theory again. For the case $\alpha = a$, suppose $P \xrightarrow{a} P'$ but there is no Q' such that $Q \xrightarrow{a} Q'$ with $\Im \vdash P' = Q'$. Take a context $C[\,] \overset{\text{def}}{=} (\overline{a}.\overline{c}.\mathbf{0} \mid c.\mathbf{0} \mid [\,])$ with c fresh. Then $C[P] \rightarrow (\overline{c}.\mathbf{0} \mid c.\mathbf{0} \mid P') \rightarrow P'$ so that there exists Q'' with $C[Q] \rightarrow^2 Q'' = P'$. Now, by the reasoning we have repeatedly used up to now, Q'' cannot contain $c.\mathbf{0}$ and $(c.\mathbf{0} \,|\, \overline{c}.\mathbf{0})$ by Lemma 6.6 together with Proposition 5.3 (ii). It follows that $Q \xrightarrow{a} Q''$ and $P' = Q''$. The case $\alpha = \bar{a}$ is symmetric to the above case. Therefore $\Im$ is a strong bisimulation, i.e. $|\Im| \subset \sim$, and, in particular, $|\Im^*_{ccs:s}| \subset \sim$. To prove the converse inclusion, we know $|\Im^*_{ccs}| = \approx \supset \sim$ from Proposition 6.8, and $\sim$ is reduction-closed and closed under reduction contexts, so $\sim$ corresponds to a strong theory with base $\Im^*_{ccs}$. Hence we have $|\Im^*_{ccs:s}| \supset \sim$. $\square$

6.10. Discussion. Compare the above result with that of [21] where the coincidence between "weak barbed congruence" for CCS and $\approx$, under the condition of image finiteness of reduction, is discussed. Note that, for both weak and strong cases, our proof is much simpler than theirs due to the introduction of the reduction-closure property. Moreover, since we do not utilise the summation operator for the observability result at all, we do not need any image-finiteness condition nor the new name postulate (i.e. the postulate that that the collection of names form a proper class) in our reasoning. For further discussions, see 7.1 and 7.2.

λ-calculus

6.11. Equational theories based on reduction and equation have been extensively studied for λ-calculus [2]. Specifically the notion of β-equality is subsumed in our reduction-closure property as noted in 3.1, not to say many of our formal constructions are inspired by λ-theories. A natural question is: can we get any (interesting) observables applying our framework based on reduction-closure to λ-calculus? Some of the obtained results are discussed in the following, hoping that they shed light both on the nature of our construction and on the equation of λ-calculus.

6.12. λ-terms and reduction. The syntax of λ-terms is classical; one first has variables, denoted by $x, y, z, \ldots$ Then we form the set of λ-terms ranged over by

$M, N, \ldots$ as follows:

$$x \mid \lambda x.M \mid MN \ .$$

$\lambda x.M$ is called a λ-abstraction, and MN is called an application. We assume the standard notion of α-equality. Reduction in this calculus is given by following rules:

$$\begin{array}{ll}
(\beta) & (\lambda x.M)N \to_\beta M\{N/x\} \\
(\xi) & M \to_\beta M' \ \Rightarrow \ \lambda x.M \to_\beta \lambda x.M' \\
(\mu) & M \to_\beta M' \ \Rightarrow \ MN \to_\beta M'N \\
(\nu) & M \to_\beta M' \ \Rightarrow \ NM \to_\beta NM' \\
(\alpha) & M \equiv_\alpha M' \to_\beta N' \equiv_\alpha N \ \Rightarrow \ M \to_\beta N
\end{array}$$

In the sequel we use the ordering due to [14, 22], which we redefine (without changing the meaning) using only the above reduction relation.

6.13. Definition. (*ordering*) M has order n, denoted $M \in \mathrm{O}_n$, if n is the largest i such that $x_1, x_2, ..., x_n \in \mathcal{V}$ and for some N, we have $M \twoheadrightarrow_\beta \lambda x_1 x_2 ... x_n.N$. M has proper order 0, denoted $M \in \mathrm{PO}_0$, if $M \in \mathrm{O}_0$ and $\neg(\exists \vec{N}.\ M \twoheadrightarrow_\beta x\vec{N})$. Such a term is also called *strongly unsolvable*. Also $\mathrm{PO}_n \stackrel{\text{def}}{=} \{M \mid \exists N \in \mathrm{PO}_0.\ M \twoheadrightarrow_\beta \lambda x_1 x_2 ... x_n.N\}$ with $n \geq 1$, and $\mathrm{PO}_\infty \stackrel{\text{def}}{=} \mathrm{O}_\infty$, we define a PO_∞-term as a term in PO_∞. We also refer to [2] as the standard text.

6.14. Sound theory. Taking the α-convertibility as base equations (note that we have no β-equality as an axiom), the notion of *reduction theories* in this setting is easily defined. Then we immediately obtain the family of reduction-closed theories (note that theories are not necesarrily allow β-equality yet), and the insensitive terms in this setting may naturally be the strongly unsolvable terms (PO_0-terms) in the above definition. However it does not seem certain that we can obtain the canonical congruence just by identifying PO_0-terms. So let us say a reduction-closed theory over λ-terms is *sound* if it equates any pair from PO_0-terms, as well as any pair from PO_∞-terms. As we shall see soon, by identifying (separately) these two types of terms with extreme behaviour, we indeed obtain $\mathcal{K}^*$, after picking up *head normal forms* as generic observables.

6.15. Incompatible pairs. We establish some incompatible pairs in reduction-closed theories with the identification proposed in 6.14, together with a fact given by Longo [14] and Ong [22]. Let $\Im, \Im', \ldots$ range over sound theories and Ins_λ be the set of insensitive λ-terms. Below we write $M \# N$ when any "sound" theory (not λ-theory) does not equate M and N.

Lemma $I \# N$ *if* $N \in \mathsf{Ins}_\lambda$.

Proof: Taking the context $C[\,] \overset{\mathrm{def}}{=} [\,]M$, suppose $P \in \mathrm{PO}_0$. Then $C[I] \to_\beta M$ but $C[P] \twoheadrightarrow_\beta N$ implies $N \equiv P'M' \in \mathrm{PO}_0$ because elements of PO_0 are closed under application (pp.29 in [22]). Thus $\Im \vdash M = P'M' = \Omega$ by the definition of sound theories, hence $\forall M.\ \Im \vdash M = \Omega$, a contradiction. Similarly for the case of PO_∞. $\square$

To establish the main result of λ-calculus, we use Wadsworth's theorem, together with Longo's important finding.

6.16. Proposition. (Observability in λ-calculus) *Suppose* $\Im$ *is a sound theory and* $\Im \vdash M = N$. *If* $M \twoheadrightarrow_\beta M'$ *and* M' *is a head normal form, then for some* N', *we have* $N \twoheadrightarrow_\beta N'$ *and* N' *is a head normal form.*

Proof: Suppose $\Im \vdash M = N$ and $M \twoheadrightarrow_\beta M'$ and M' is a head normal form, but there is no N' such that $N \twoheadrightarrow_\beta N'$ and N' is a head normal form. By the reduction-closed property, there should be N' such that $N \twoheadrightarrow_\beta N'$ and $\Im \vdash M' = N'$. Since M' is a head normal form, $\exists z_1 \cdots z_j,\ R_1 \cdots R_m$ such that $(\lambda z_1 ... z_j.M')R_1 \cdots R_m \twoheadrightarrow_\beta I$. Now we take a context $C[\,] \overset{\mathrm{def}}{=} (\lambda z_1 ... z_j.[\,])R_1 \cdots R_m$ with the above $z_1 \cdots z_j,\ R_1 \cdots R_m$. Then we have $C[M'] \twoheadrightarrow_\beta I$.

To reason about syntactic behavior of λ-terms, the division of unsolvable terms into two species by Longo (Lemma 1.4 of [14]) becomes crucial. Suppose M is unsolvable. Then we have either:

(1) $M \in \mathrm{PO}_\infty$, or
(2) $\exists n \geq 0. \exists N \in \mathrm{PO}_0.\ M \twoheadrightarrow_\beta \lambda x_1 x_2 .. x_n.N$.

According to this dichotomy, we have two cases.

Suppose $N \in \mathrm{PO}_\infty$. Then it is obvious that $\lambda z_1 ... z_j.N' \in \mathrm{PO}_\infty$, then for all N'' such that $C[N'] \twoheadrightarrow_\beta N''$ implies $N'' \in \mathrm{PO}_\infty$ because PO_∞ is closed under application. Thus $\Im \vdash I = L$ for some $L \in \mathrm{PO}_\infty$, a contradiction by Lemma 6.15.

Next we assume $N' \twoheadrightarrow_\beta \lambda y_1 .... y_i.Q$ and $Q \in \mathrm{PO}_0$. Let $j+i = n$ and $z_1, ..., z_j, y_1, ..., y_i \overset{\mathrm{def}}{=} x_1, ..., x_n$. There are two sub cases. When $n \leq m$, we take the above context $C[\,]$. Then

$$C[N'] \twoheadrightarrow_\beta (Q\{R_1/x_1\}\{R_2/x_2\}...\{R_n/x_n\})R_{n+1} \cdots R_m \twoheadrightarrow_\beta N'' \in \mathrm{PO}_0$$

since Q is closed under substitution, application and reduction. Hence $\Im \vdash I = N''$, which contradicts by Lemma 6.15. In the case $n > m$, take the context $C'[\,] \overset{\mathrm{def}}{=} C[\,]\vec{I}$ where $C[\,]$ is the above context and $|\vec{I}| = n - m$. Then $C'[I] \twoheadrightarrow_\beta I$, but

$$C'[N'] \twoheadrightarrow_\beta Q\{R_1/x_1\}\{R_2/x_2\}...\{R_m/x_m\}\{I/x_{m+1}\}..\{I/x_n\} \in \mathrm{PO}_0$$

which implies, by the same argument as in the previous case, for all N'' such that $C'[N'] \twoheadrightarrow_\beta N''$, then N'' is an element of PO_0. Hence a contradiction by Lemma 6.15. Hence we conclude the above proposition. □

Finally we show:

6.17. Corollary. *The maximum sound theory coincides with $\mathcal{K}^*$, the theory of D_∞.*

Proof: First Proposition 6.16 is equivalent to, for some sound theory $\Im$,

$\Im \vdash M = N \;\; \Rightarrow$

$\forall C.(C[M]$ has some head normal form $\;\Leftrightarrow\;$ $C[N]$ has some head normal form).

Thus we can get $\Im \vdash M = N \;\Leftrightarrow\; \forall C.(C[M]$ solvable $\Leftrightarrow$ $C[N]$ solvable), which tells us that the family of consistent sound theories are strictly included in $\mathcal{K}^*$. But it is immediate that $\mathcal{K}^*$ is a sound reduction theory because all Ins_λ-elements are unsolvable and because, by β-equality, the reduction-closure property can be obtained vacuously. Hence the result. □

Note that, with this canonical theory, we have restored *β-equality*, which would reinforce our standpoint to regard β-equality as a special case of the reduction-closure property (cf. 3.1). Explication of the development in this line, specifically the reason of the necessity to identify PO_∞-terms in addition, is left to our future report.

6.18. Question. What may be the result of applying the "strong" construction of Section 5 to (sound) theories in this setting, specifically to $\mathcal{K}^*$ (cf. [13])?

7. Concluding Remarks

7.1. Comparison with convergence-based semantics for processes. There are two important predecessors to our work. Boudol [4] studied (in)equation over essentially the same system as the ν-calculus based on Morris-like contextual precongruence. Milner and Sangiorgi [21] have shown that their construction called the *barbed congruence* coincides with strong bisimilarity in the case of CCS, also referring to the results in the weak case. An apparent difference is that we do not positively stipulate a presupposed notion of convergence to derive a canonical equality. Nevertheless we note that our approach, a posteriori, reveals the import of the convergence predicate, see e.g. Proposition 3.24, by *discovering* such a notion without stipulating it beforehand.

7.2. Reduction-closure in Milner-Sangiorgi's construction. One important difference from [21] lies in our crucial usage of the reduction-closure property. While [21] uses a closure property on the reduction relation, the property is applied to usual (i.e. not necessarily congruent) relations over terms in general together with the convergence predicate. One then takes the maximum congruence within the union of all such relations. In contrast, our formulation regards the closure property as the essential criterion for *equality* over stateful processes, just as convertibility is the criterion for equality over terms denoting functions. While taking the maximum congruence within a non-congruent equivalence is already known in the case of observation congruence in CCS [16], we would note that the situation in [21] is quite different from [16]; the base relation $\approx$ in [16] is behaviourally sound while this may not be the case in barbed bisimilarity in [21] (where $a.0$ and $b.0$ are equated). Thus, we cannot know, at least *a priori*, whether taking the congruence within it is significant or not.

7.3. Comparison with sensible theories: (1) equational properties. We have already suggested how our semantic scheme parallels the one in sensible λ-theories. More concretely, we have:

- reduction-closure property vs. β-equality as the basic equality, and
- insensitive terms vs. unsolvable terms as the notion of meaningless terms.

Note that $\mathcal{K}^*$ [2] is the maximum sensible λ-theory and a maximal λ-theory, just as $\Im^*_\nu$ is the maximum sound ν-theory and a maximal reduction-closed ν-theory (such a maximal theory is called *relatively Hilbert-Post complete* with respect to a concerned family of theories in [22]). Moreover, just as $\Im^*_\nu$ has interesting theories like $\Im_{\approx_s}$ and $\Im_{\approx_a}$ as its (sound) subtheories, $\mathcal{K}^*$ has interesting λ-theories like the theory of the Böhm tree $\mathfrak{B}$ and Morris's theory, as its (sensible) subtheories (so, e.g. $\Im^*_\nu$ is a unique Hilbert-Post completion of $\Im_{\approx_s}$ w.r.t. reduction-closed ν-theories, in the terminology of [22]). However notice that $\mathcal{K}^*$ is also maximal among the family of all consistent equational theories over λ-terms, that is, it is *Hilbert-Post complete* in the sense of [2]. In this respect, $\mathcal{K}^*$ has a stronger equational property than $\Im^*_\nu$.

7.4. Comparison with sensible theories: (2) insensitive terms. The identification of insensitive terms in sound ν-theories functions quite similarly to that of unsolvable terms in sensible λ-theories. In retrospect, a crucial fact turns out to be that the soundness characterises a subfamily of the power set of $\Im^*_\nu$ which contains $\Im^*_\nu$; this immediately gives us the isolation sets which are essentially equivalence classes induced by $\Im^*_\nu$, just as sensible λ-theories [2]. However the analogy breaks down here; in sound theories, insensitive terms themselves are *not* isolated (unlike unsolvable terms in sensible theories), e.g. $\Im^*_\nu \vdash \mathsf{I}(a) = \Lambda$ [because: $\approx_a \subset |\Im^*_\nu|$ by 3.16]. While this is consistent with our intention of starting from the smallest semantic commitment to obtain the broadest meaningful equality over processes, one may find it interesting whether some syntactic characterisation of the "semantic" insensitive terms, i.e. $\{P \mid \Im_\nu^* \vdash P = \Lambda\}$, which in many senses correspond to unsolvable λ-terms in the setting of ν-theories, exists or not.

7.5. Open issues. In the following, we list some of naturally arising open issues.

- Can we have a reduction-based semantic formulation which can derive process equivalences other than bisimilarities, cf. [1]? As far as *may* and *must* equivalences [5] go, a possible candidate is the set of "semantic" insensitive terms referred to at the end of the preceding subsection, combined with a certain divergence notion. But this is rather indirect (since it uses $\Im_\nu^*$, i.e. via reduction-closure).
- Milner asked what may be the general condition under which the (weak) equality induced by the construction in [21] and that induced by the construction here coincide. Apart from the usage of insensitive terms in the present framework, this asks the following question: when, under the same observability condition, do the reduction closure for equality and that for equivalence, from which we take the maximum congruence, coincide? By answering such a question, we may obtain deeper understanding on the reduction closure itself.
- The similarity in the way of deriving canonical congruences between our construction and those found in λ-calculi [2, 1, 22] is remarkable as was already noted. What is common between these constructions is that the canonical equality is obtained as the maximum element of a sizable family of "meaningful" equalities which equate insignificant terms. The method is not only mathematically elegant but also provides a tractable proof method (cf. 4.3–4.4). Mathematical canonicity may suggest correspondence with important models of the concerned formalism; at least such are cases in strict and lazy λ-theories, though concrete results in the concurrency setting are still missing. Such correspondence may also be helpful in clarifying the semantic properties shared by those formalisms to which the approach based on the identification of behaviourally insignificant terms are applicable. One may ask: can we use $\Im_\nu^*$ as a clue to the construction of a syntax-free model of the ν-calculus?

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APPENDIX A. PROOF OF PROPOSITION 2.13

A.1. Congruence of $\approx_s$. Regarding (i), we first show the main line of the reasoning to establish the congruence of $\approx_s$ since it can serve as the representative of other cases. We then note what are differences for congruence of $\approx_a$, $\sim_s$, and $\sim_a$. For $\approx_s$ we shall verify the following statements: of which (b) to (g) testify the congruence of $\approx_s$. We note that, other than (b), (e) and (f), the reasoning is quite similar to what has been given in the context of ground bisimilarity for π-calculus in [20].

(a) (variant lemma) $P \approx_s Q \;\Rightarrow\; P\{b/a\} \approx_s Q\{b/a\}$ if $b \notin \mathcal{FN}(P, Q)$.

(b) $\approx_s$ is an equivalence relation.

(c) $P \approx_s Q \;\Rightarrow\; c \blacktriangleright P \approx_s c \blacktriangleright Q$.

(d) $P \approx_s Q \;\Rightarrow\; P, R \approx_s Q, R$.

(e) $P \approx_s Q \;\Rightarrow\; ax.P \approx_s ax.Q$.

(f) $P \approx_s Q \;\Rightarrow\; !ax.P \approx_s !ax.Q$.

(g) $P \approx_s Q \;\Rightarrow\; P\{v/x\} \approx_s Q\{v/x\}$ for any x and v.

(a). We show the union of the following relation is a bisimulation.

$$\begin{array}{llcl}
\mathcal{R}_0 : & \approx_s & & \\
\mathcal{R}_1 : & P\,\mathcal{R}_0\,Q & \Rightarrow & P\{c'/c\}\,\mathcal{R}_1\,Q\{c'/c\} \;\wedge\; c' \notin \mathcal{FN}(P, Q) \\
 & \vdots & \vdots & \\
\mathcal{R}_{n+1} : & P\,\mathcal{R}_n\,Q & \Rightarrow & P\{c'/c\}\,\mathcal{R}_{n+1}\,Q\{c'/c\} \;\wedge\; c' \notin \mathcal{FN}(P, Q)\ .
\end{array}$$

To check this, we use induction on n and prove: if $P\mathcal{R}_nQ$ then whenever $P \xrightarrow{l}_s P'$ which is relevant to Q, we have $Q \overset{\hat{l}}{\Longrightarrow}_s Q'$ with $P'\mathcal{R}_{n+1}Q'$, noting $\mathcal{R}_{n+1} \supset \mathcal{R}_n$. The base case is trivial. For the inductive step, we consider the case $l = \uparrow a(b)$, which is the most difficult. Suppose $P\mathcal{R}_{n-1}Q$ for which the condition holds. We think about a pair $P\{c'/c\}$ and $Q\{c'/c\}$ with $c' \notin \mathcal{FN}(P, Q)$ (so $P\{c'/c\}\mathcal{R}_nQ\{c'/c\}$).

Assume $P\{c'/c\} \xrightarrow{\uparrow a(b)}_s P'$. Let b' be fresh and write $l_1, l_2, \sigma, \sigma^{-1}, \rho$ and ρ^{-1} for $\uparrow a(b), \uparrow a(b'), \{c'/c\}, \{c/c'\}, \{b'/b\}$ and $\{b/b'\}$. Below $l_2\sigma^{-1}$ denotes $\uparrow a\{c/c'\}(b')$. Then:

$$\begin{array}{lcll}
P\sigma \xrightarrow{l_1}_s P' & \Rightarrow & P\sigma \xrightarrow{l_2}_s P'\rho & (1) \\
 & \Rightarrow & P \xrightarrow{l_2\sigma^{-1}}_s P'\rho\sigma^{-1} & (2) \\
 & \Rightarrow & Q \overset{l_2\sigma^{-1}}{\Longrightarrow}_s Q''\mathcal{R}_n(P'\rho\sigma^{-1}) & (3) \\
 & \Rightarrow & Q\sigma \overset{l_2}{\Longrightarrow}_s Q''\sigma & (4) \\
 & \Rightarrow & Q\sigma \overset{l_1}{\Longrightarrow}_s (Q''\sigma\rho^{-1})\mathcal{R}_{n+1}P' & (5)
\end{array}$$

(1), (2) and (4) are by easy syntactic reasoning, while (3) is by the inductive hypothesis and the relevance of $l_2\sigma^{-1}$ to Q, and (5) is by the relevance of l_1 to Q as well as because $Q''\sigma$ cannot contain b. Other cases are easy.

(b). The symmetry and reflexivity of $\approx_s$ are obvious. For transitivity, we show:

$$\begin{array}{llcl}
\mathcal{R}_0 : & \approx_s\approx_s & & \\
\mathcal{R}_1 : & P\ \mathcal{R}_0\ Q & \Rightarrow & P\{c'/c\}\ \mathcal{R}_1\ Q\{c'/c\}\ \wedge\ c' \notin \mathcal{FN}(P,Q) \\
 & \vdots & \vdots & \\
\mathcal{R}_{n+1} : & P\ \mathcal{R}_n\ Q & \Rightarrow & P\{c'/c\}\ \mathcal{R}_{n+1}\ Q\{c'/c\}\ \wedge\ c' \notin \mathcal{FN}(P,Q)
\end{array}$$

is a bisimulation, showing $P\mathcal{R}_nQ$ and $P \xrightarrow{l}_s P'$ implies $Q \overset{l}{\Longrightarrow}_s Q'\mathcal{R}_{n+1}P'$ if l is relevant to Q. We show the base case where the transition is a bound output. Suppose $P \approx_s R \approx_s Q$ and $P \xrightarrow{\upharpoonright a(b)}_s P'$ with $b \notin \mathcal{FN}(Q)$. Then, assuming c is fresh, we easily get $Q \xrightarrow{\upharpoonright a(c)}_s Q''$ with $P'\{c/b\} \approx_s\approx_s Q''$, hence we get $Q \xrightarrow{\upharpoonright a(b)}_s Q''\{b/c\}\mathcal{R}_1P'$. Other cases are easy. The inductive step is verified with a reasoning essentially identical with that of Claim (a).

(c). We show $\mathcal{R} \overset{\text{def}}{=} \{\langle c \blacktriangleright P\,,\, c \blacktriangleright Q\rangle \mid P \approx_s Q\} \cup \approx_s$ is a bisimulation by checking the transition. For two kinds of output and τ transition, the reasoning is easy. For input, if $c \blacktriangleright P \xrightarrow{\downharpoonleft ab}_s P'$, then we know that we can write, with fresh c', $c' \blacktriangleright (P\{c'/c\}) \xrightarrow{\downharpoonleft ab}_s P' \equiv_\alpha c' \blacktriangleright P''$ hence $P\{c'/c\} \xrightarrow{\downharpoonleft ab}_s P''$. But by Claim (a), $Q\{c'/c\} \overset{\downharpoonleft ab}{\Longrightarrow}_s Q'' \approx_s P''$ from which $c' \blacktriangleright Q \overset{\downharpoonleft ab}{\Longrightarrow}_s c' \blacktriangleright Q''$ is easy.

(d). We show $\mathcal{R} \overset{\text{def}}{=} \{\langle \tilde{c} \blacktriangleright (P,R)\,,\, \tilde{c} \blacktriangleright (Q,R)\rangle \mid P \approx_s Q\}$ is a bisimulation by syntactic reasoning.

(e). We show $\mathcal{R} \overset{\text{def}}{=} \{\langle ax.P\,,\, ax.Q\rangle \mid P \approx_s Q\} \cup \approx_s$ is a bisimulation. The only case to be considered is input, and this is easy from the proviso concerning bisimilarity over open terms.

(f). Similarly we show $\mathcal{R} \overset{\text{def}}{=} \{\langle !ax.P\,,\, !ax.Q\rangle \mid P \approx_s Q\} \cup \approx_s$ is a bisimulation, easy as in Claim (e).

(g) is again direct from the definition of bisimilarity over open terms. □

A.2. Congruence of other bisimilarities. To prove congruence of other bisimilarities, essentially the same techniques can be used. Some details are however different, which are listed below.

- *The case for* $\sim_s$. The reasoning exactly follows that of $\approx_s$.

- *The case for* $\approx_a$. The reasoning basically follows that of $\approx_s$. To prove what corresponds to Claim (e) above, however, we rather use the following: with R being an arbitrary sequence of concurrent composition of messages, $\mathcal{R} \overset{\text{def}}{=} \{\langle (ax.P, R)\,,\,(ax.Q, R)\rangle \mid P \approx_a Q\} \cup \approx_a$ is an asynchronous weak bisimulation.
- *The case for* $\sim_a$. Similar to $\approx_a$.

A.3. Inclusions and incompatibility. Regarding (ii), i.e. inclusions and incompatibility results, we first show the inclusion. This we establish by proving that $\approx_s$ (resp. $\sim_s$) is a weak (resp. strong) asynchronous bisimulation. Since output and τ transitions coincide between $\approx_s$ and $\approx_a$ (resp. $\sim_s$ and $\sim_a$), the only remaining case is the input transition. But this is immediate from the congruence of $\approx_s$ (resp. $\sim_s$). Finally it is easy to check the proper inclusions and incompatibility, using the pairs listed at the end of 2.13.

APPENDIX B. PROOF OF PROPOSITION 4.3

We prove:

(a) $\Im_{\mathsf{EQ}}$ *is reduction-closed, and*

(b) $\Im_{\mathsf{EQ}}$ *isolates a set* $\{P \mid\ P \twoheadrightarrow \overset{\uparrow a}{\rightsquigarrow}\}$ *for each* a.

As discussed in the main section, the proposition holds if these are established.

For both (a) and (b), we use the following notion of *equations over names derived from a series of equators.* Let $\prod_i^n P_i$ stand for $P_1, P_2, ..., P_n$ (if $n = 0$, this denotes Λ). Then $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v$ or just $u \frown v$ with $\prod_i^n \mathsf{EQ}(u_i v_i)$ implicit, if it is derived from the following.

- $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v \quad$ if $u \overset{\text{def}}{=} v$ or $u = u_i \ \wedge\ v = v_i$ for some i.
- $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v \ \Rightarrow\ \prod_i^n \mathsf{EQ}(u_i v_i) \vdash v \frown u$.
- $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v \ \wedge\ \prod_i^n \mathsf{EQ}(u_i v_i) \vdash v \frown w \ \Rightarrow\ \prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown w$.

Then $|\prod_i^n \mathsf{EQ}(u_i v_i)|$ is a set of equations which are deducible from $\prod_i^n \mathsf{EQ}(u_i v_i)$. Similarly we write: $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash P \frown Q$ when $P \overset{u_0 \cdot u_1}{\leftrightarrow} P_1, ..., P_n \overset{u_{n-1} \cdot u_n}{\leftrightarrow} Q$ $(n \geq 1)$ with, for each i, $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u_{i-1} \frown u_i$.

Let the theory $\Im_{\mathsf{EQ}}^-$ be defined as a theory with the same set of axioms as $\Im_{\mathsf{EQ}}$ and, as rules, those in Definition 2.15 except rules (7) and (8). For this theory, it is easy to know, by induction on the derivation of equations, that $\Im_{\mathsf{EQ}}^- \vdash P = Q$ if and only if $P \equiv \tilde{c} \blacktriangleright (P', \prod_i^n \mathsf{EQ}(u_i v_i))$ and $Q \equiv \tilde{c} \blacktriangleright (Q', \prod_i^n \mathsf{EQ}(u'_i v'_i))$ where:

- $|\prod_i^n \mathsf{EQ}(u_i v_i)| = |\prod_i^n \mathsf{EQ}(u'_i v'_i)|$.
- P' and Q' are zero or more parallel composition of receptors, replications, and messages.
- $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash P' \frown Q'$.

Now we prove (a). For the purpose we show $\Im^{-}_{\mathsf{EQ}}$ (cf. 3.5) is reduction-closed, which, together with Lemma 3.6, tells us that $\Im_{\mathsf{EQ}}$ itself is reduction-closed. But if $\Im^{-}_{\mathsf{EQ}} \vdash P = Q$ and $P \rightarrow R$, then, writing $P \equiv \tilde{c} \blacktriangleright (P', \prod_i^n \mathsf{EQ}(u_i v_i))$ and $Q \equiv \tilde{c} \blacktriangleright (Q', \prod_i^n \mathsf{EQ}(u'_i v'_i))$, *either*

(i) $R \equiv \tilde{c} \blacktriangleright (P'', \prod_i^n \mathsf{EQ}(u_i v_i))$ with $P' \rightarrow P''$, *or*

(ii) $R \equiv \tilde{c} \blacktriangleright (P'', \leftarrow uw, \prod_i^n \mathsf{EQ}(u_i v_i))$ with $P' \equiv (P'', \leftarrow vw)$, where $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v$.

(ii) is easy since $\Im^{-}_{\mathsf{EQ}} \vdash P = R$. For (i), we can write, assuming x is utterly fresh except in S, *either*

- $P' \equiv (ux.S, \leftarrow uw, T)$ with $P'' \equiv (S\{w/x\}, T)$, *or*
- $P' \equiv (!ux.S, \leftarrow uw, T)$ with $P'' \equiv (S\{w/x\}, !ux.S, T)$.

For the former, we have: $Q' \equiv (u'x.S', \leftarrow v'w', T')$ with (under $\prod_i^n \mathsf{EQ}(u_i v_i)$) $u \frown u'$, $u \frown v'$, $w \frown w'$, $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash S \frown S'$, and $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash T \frown T'$. But then

$$\leftarrow v'w', \prod_i^n \mathsf{EQ}(u_i v_i) \twoheadrightarrow \leftarrow u'w', \prod_i^n \mathsf{EQ}(u_i v_i)$$

hence done. The latter is also easy, writing $Q' \equiv (!u'x.S', \leftarrow v'w', T')$ with the same condition as the former case.

For (b), we use the property similar to Lemma 3.6 for the required isolation, i.e. we show that, if $\Im^{-}$ (understood as in 3.5) isolates each set of the form $\{P \mid P \twoheadrightarrow \overset{\upharpoonleft a}{\rightsquigarrow}\}$, then $\Im$ also does so, similarly to Lemma 3.6 (use the result stated in Remark 3.7 with $P\mathcal{R}Q$ defined as: for any a, $(P \twoheadrightarrow \overset{\upharpoonleft a}{\rightsquigarrow} \wedge Q \twoheadrightarrow \overset{\upharpoonleft a}{\rightsquigarrow}) \vee (\neg(P \twoheadrightarrow \overset{\upharpoonleft a}{\rightsquigarrow}) \wedge \neg(Q \twoheadrightarrow \overset{\upharpoonleft a}{\rightsquigarrow})))$. Therefore if we show that $\Im^{-}_{\mathsf{EQ}}$ has the above isolation sets, then we are done. But this is easy by a property of (a chain of) equators in the following:

$$\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v \quad \Rightarrow \quad \prod_i^n \mathsf{EQ}(u_i v_i), \leftarrow uw \twoheadrightarrow \prod_i^n \mathsf{EQ}(u_i v_i), \leftarrow vw\ .$$

The property itself is easily proved by induction on the length of derivation of $\prod_i^n \mathsf{EQ}(u_i v_i) \vdash u \frown v$. □

APPENDIX C. PROOF FOR PROPOSITION 4.9 (II)

The relation $\mathcal{S}$ is defined as:

$$\mathcal{S} \stackrel{\text{def}}{=} \{\langle P,\ Q\rangle \mid \Im^{*}_{\nu=} \vdash C_n[P] = C_n[Q]\}$$

with $C_n[\,] \stackrel{\text{def}}{=} \tilde{e} \blacktriangleright ([\,], \prod_i^n \mathsf{EQ}(e_i d_i), \prod_i^n !\mathsf{SW}(c_i e_i))$, for some n, and where it should be the case that

- $\tilde{e} = e_1..e_n$ are all distinct, and
- $c_1..c_n$ and $d_1..d_n$ are all fresh and distinct.

We easily know that $|\Im^*_{\nu=}| \subset \mathcal{S}$. The crucial point regarding the context $C_n[\ ]$ is that, by only observing the behaviour of $C_n[P]$, where names in $\tilde{c}$ and $\tilde{d}$ do not occur in P, we can uncover the behaviour of P, mainly using incompatible pairs. We also use the following expressions as auxiliary agents.

(i) $\mathtt{mvc}(uvw) \overset{\mathrm{def}}{=} ux.([x = v] \leftarrow w),\ w.\Lambda$ ("mvc" means a value checker with the match operator).

(ii) $\mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(R) \overset{\mathrm{def}}{=} (\prod_i^n (\leftarrow c_i g_i),\ g_1 x_1.g_2 x_2 \ldots g_n x_n.(R,\ \leftarrow e),\ e.\Lambda)$.

(iii) $S(a\tilde{x}de\tilde{f}d_{n+1}c_{n+1}) \overset{\mathrm{def}}{=}$

$$ax.(\textstyle\prod_i^m ([x = f_i] \leftarrow e), \prod_i^n ([x = x_i] \leftarrow e), \mathsf{EQ}(xd_{n+1}), !\mathsf{SW}(c_{n+1}x), \leftarrow d), d.\Lambda.$$

The following shows how $C_n[\,]$ as well as the above expressions can be used. Below we assume, w.l.o.g. that whenever there is a transition or reduction from $C_n[P]$ reaching a certain term, that reduction/transition does not alter names in $\tilde{e}$ by α-conversion.

Notation: In the following, we write $P = Q$ for $\Im^*_{\nu=} \vdash P = Q$.

Claim Let $C_n[\ ]$ be the context as defined above, with specified conditions on its names. Let a and b be names not included in $\{\tilde{d}\}$ (actually they cannot be in $\tilde{e}$ or $\tilde{c}$ either).

(1) If $C_n[P] \twoheadrightarrow R$, then $\exists P'.\ (R = C_n[P'] \wedge P \twoheadrightarrow P')$.

(2) If $C_n[P] \overset{\uparrow ab}{\Longrightarrow}_a R$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\uparrow ab}{\Longrightarrow}_a P')$.

(3) If $C_n[P] \overset{\uparrow d_i b}{\Longrightarrow}_a R$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\uparrow e_i b}{\Longrightarrow}_a P')$.

(4) If $(C_n[P], \mathsf{SW}(ac), \mathtt{mvc}(d_j cc')) \twoheadrightarrow R$ with c, c' fresh and $c' \notin \mathcal{AN}(R)$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\uparrow a e_j}{\Longrightarrow}_a P')$.

(5) If $(C_n[P], \mathsf{SW}(d_i c), \mathtt{mvc}(d_j cc')) \twoheadrightarrow R$ with c, c' fresh and $c' \notin \mathcal{AN}(R)$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\uparrow e_i e_j}{\Longrightarrow}_a P')$.

(6) If $(C_n[P], \leftarrow ab) \twoheadrightarrow R$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\downarrow ab}{\Longrightarrow}_a P')$.

(7) With $e, g_1, \ldots, g_n$ are all fresh and pairwise distinct, and $x_1, x_2, \ldots, x_n$ are pairwise distinct, we have: $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(R_0)) \twoheadrightarrow R$ with $e \notin \mathcal{AN}(R)$, then $\exists P'.\ (R = C_n[P'] \ \wedge\ (P, R_0\{\tilde{e}/\tilde{x}\}) \twoheadrightarrow P')$.

(8) With the same condition on names as in (7), if $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(\leftarrow x_i b)) \twoheadrightarrow R$ and $e \notin \mathcal{AN}(R)$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\downarrow e_i b}{\Longrightarrow}_a P')$.

(9) With the same condition on names as in (7), if $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(\leftarrow a x_j)) \twoheadrightarrow R$ and $e \notin \mathcal{AN}(R)$, then $\exists P'.\ (R = C_n[P'] \wedge P \overset{\downarrow a e_j}{\Longrightarrow}_a P')$.

(10) With the same condition on names as in (7), if $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(\leftarrow x_i x_j)) \twoheadrightarrow R$ and $e \notin \mathcal{AN}(R)$, then $\exists P'. (R = C_n[P'] \wedge P \overset{\downarrow e_i e_j}{\Longrightarrow}_a P')$.

(11) Let $\mathcal{FN}(C_n[P]) \subset \{f_1, .., f_m\}$, $\tilde{x} = x_1..x_n$, and $d_{n+1}, c_{n+1}, e', e, d$ fresh. Then, writing just $S(a\tilde{x})$ for $S(a\tilde{x}de\tilde{f}d_{n+1}c_{n+1})$, if we have: $(C_n[P],\ \mathtt{in}^{e'}_{\tilde{c}\tilde{x}\tilde{g}}(S(a\tilde{x})))$ $\twoheadrightarrow R$ where $e', d, e \notin \mathcal{AN}(R)$, then: $R = C_{n+1}[P']$ with $P \overset{\uparrow a(e_{n+1})}{\Longrightarrow}_a P'$ where e_{n+1} is fresh except in P', and

$$C_{n+1}[\,] \overset{\text{def}}{=} \tilde{e}e_{n+1} \blacktriangleright([\,],\ \prod_i^{n+1} \mathsf{EQ}(e_i d_i),\ \prod_i^{n+1} !\mathsf{SW}(c_i e_i)).$$

(12) If $(C_n[P],\ \mathtt{in}^{e'}_{\tilde{c}\tilde{x}\tilde{g}}(S(d_i\tilde{x}))) \twoheadrightarrow R$ and $e', d, e \notin \mathcal{AN}(R)$, then we have: $R = C_{n+1}[P']$ with $P \overset{\uparrow e_i(e_{n+1})}{\Longrightarrow}_a P'$ where e_{n+1} is fresh expect in P' and $C_{n+1}[\,]$ is a context defined in (11).

Below "(i)" denotes the first part of this proposition, and (1), (2), .. are clauses in the Claim.

Proof of Claim: By mechanical syntactic inspection using incompatible pairs and that $[a = b]P = \Lambda$. Below we only show the proofs of two important cases, (1) and (11). For other cases (where we use (1) throughout): (2) uses (i), while (3) and (7) are easy. (4) and (5) are established with (i) and (3). (6) is trivial by $C_n[P], \leftarrow ab \equiv C_n[P, \leftarrow ab]$, and (8)–(10) use (6) and (7). (12) is just similar to (11) with (3).

Regarding (1), we note that the interaction between P and $C_n[\,]$ always use equators. But then a message which interacts with an equator can be "rolled back" in the way

$$(\leftarrow e_i b,\ \mathsf{EQ}(e_i d_i)) \ \rightarrow\ (\leftarrow d_i b,\ \mathsf{EQ}(e_i d_i)) \ \rightarrow\ (\leftarrow e_i b,\ \mathsf{EQ}(e_i d_i))\ .$$

Now, if $P \twoheadrightarrow Q$ and $Q \twoheadrightarrow P$, we always get $P = Q$ since the set of such pairs easily form an asynchronous weak bisimulation. We also know, by induction on the structure of terms as well as on the number of reduction steps:

$$(P, \mathsf{EQ}(ab)) \twoheadrightarrow R \quad \Rightarrow \quad R \twoheadrightarrow (P', \mathsf{EQ}(ab)) \ \wedge\ P \twoheadrightarrow P'$$

Together with the first fact, the result is easily obtained.

Regarding (11), since e' does not occur actively in R, with (7) it should be the case that

$$(C_n[P],\ \mathtt{in}^{e'}_{\tilde{c}\tilde{x}\tilde{g}}(S(a\tilde{x}))) \ \twoheadrightarrow\ C_n[P', S(a\tilde{e})] \ \twoheadrightarrow\ R$$

with $P \twoheadrightarrow P_0' \wedge C_n[P'] = C_n[P_0']$ as in (1). Due to the lack of free d in R we should have:

$$\begin{array}{ll}
\multicolumn{2}{l}{C_n[P', S(a\tilde{e})]} \\
\twoheadrightarrow & C_n[h' \blacktriangleright (P'', \leftarrow ah'), S(a\tilde{e})] \\
\rightarrow & C_n[h' \blacktriangleright (P'', \prod_i^m([h' = f_i] \leftarrow e), \prod_i^n([h' = e_i] \leftarrow e),\ \mathsf{EQ}(h' d_{n+1}),\ !\mathsf{SW}(c_{n+1} h')),\ \leftarrow d,\ d.\Lambda] \\
\rightarrow & C_n[h' \blacktriangleright (P'', \prod_i^m([h' = f_i] \leftarrow e), \prod_i^n([h' = e_i] \leftarrow e),\ \mathsf{EQ}(h' d_{n+1}),\ !\mathsf{SW}(c_{n+1} h'))] \\
\twoheadrightarrow & R
\end{array}$$

where h' is not in $\tilde{f}\tilde{e}\tilde{d}\tilde{c}$ (note if it were in $\tilde{f}$ or in $\tilde{e}$, we would have $P \overset{\uparrow e}{\leadsto}$, contradiction; other cases can be avoided because the name comes from a message in P'). Also, w.l.o.g., in light of (1) above, we assume P'' does not contain any name from $\tilde{d}d_{n+1}$, which lets us have $P \overset{\uparrow a(h')}{\Longrightarrow}_a P''$. By putting $h' = e_{n+1}$, and observing that $\prod_i^n([e_{n+1} = f_i] \leftarrow e)$ and $\prod_i^n([e_{n+1} = e_i] \leftarrow e)$ are insensitive, we know that, if

$$C_n[e_{n+1} \blacktriangleright (P'', \prod_i^n([e_{n+1} = f_i] \leftarrow e), \prod_i^n([e_{n+1} = e_i] \leftarrow e),\ \mathsf{EQ}(e_{n+1} d_{n+1}),\ !\mathsf{SW}(c_{n+1} e_{n+1}))] \twoheadrightarrow R\ ,$$

then we have $R = C_{n+1}[P']$ with $P \overset{\uparrow a(e_{n+1})}{\Longrightarrow}_a P'$ (the final step uses (1)). □

Using the above, we now establish the bisimulation of $\mathcal{S}$. Below, as before, "(i)" denotes the first part of this proposition. (1), (2), .. are clauses in the Claim.

Case $P \xrightarrow{\tau}_a P'$. Then:

$$\begin{array}{lll}
C_n[P] \rightarrow C_n[P'] & \Rightarrow & C_n[Q] \twoheadrightarrow R = C_n[P'] \quad \text{(reduction-closure)} \\
 & \Rightarrow & R = C_n[Q'] \wedge Q \twoheadrightarrow Q' \quad (1)
\end{array}$$

Case $P \xrightarrow{\uparrow ab}_a P'$. Then:

$$\begin{array}{lll}
C_n[P] \xrightarrow{\uparrow ab}_a C_n[P'] & \Rightarrow & C_n[Q] \overset{\uparrow ab}{\Longrightarrow}_a R = C_n[P'] \quad \text{(i)} \\
 & \Rightarrow & R = C_n[Q'] \wedge Q \overset{\uparrow ab}{\Longrightarrow}_a Q' \quad (2)
\end{array}$$

Case $P \xrightarrow{\uparrow e_i b}_a P'$. Then $C_n[P] \rightarrow\xrightarrow{\uparrow d_i b}_a C_n[P']$ and the rest is similar to the above, using (3) instead of (2).

Case $P \xrightarrow{\uparrow a e_j}_a P'$. Then, with c, c' fresh,

$$\begin{array}{lll}
\multicolumn{3}{l}{(C_n[P], \mathsf{SW}(ac), \mathtt{mvc}(d_j c c')) \twoheadrightarrow C_n[P']} \\
\Rightarrow & (C_n[Q], \mathsf{SW}(ac), \mathtt{mvc}(d_j c c')) \twoheadrightarrow R = C_n[P'] & \text{(reduction-closure)} \\
\Rightarrow & R = C_n[Q'] \ \wedge\ Q \overset{\uparrow a e_j}{\Longrightarrow}_a Q' & (4)
\end{array}$$

Case $P \xrightarrow{\uparrow e_i e_j}_a P'$. Then $(C_n[P], \mathsf{SW}(d_i c), \mathtt{mvc}(d_j c c')) \twoheadrightarrow C_n[P']$ and the rest is similar to the above, using (5) instead.

Case $P \xrightarrow{\downarrow ab}_a P'$. Straightforward from (6).

Case $P \xrightarrow{\downarrow e_i b}_a P'$. Straightforward from (8).

Case $P \xrightarrow{\downarrow a e_j}_a P'$. Straightforward from (9).

Case $P \xrightarrow{\downarrow e_i e_j}_a P'$. Straightforward from (10).

Case $P \xrightarrow{\uparrow a(b)}_a P'$. Let us write $S(a\tilde{x})$ for $S(a\tilde{x}de\tilde{f}d_{n+1}c_{n+1})$, with $\{f_1..f_n\} = \mathcal{FN}(C_n[P]) \cup \mathcal{FN}(C_n[Q])$. Then, by the construction, we have:

$(C_n[P], \mathtt{in}^{e'}_{\tilde{c}\tilde{x}\tilde{g}}(S(a\tilde{x}))) \twoheadrightarrow C_n[P, S(a\tilde{e})] \rightarrow C_{n+1}[P', \prod_i^m([b = f_i] \leftarrow e), \prod_i^n([b = e_i] \leftarrow e)] = C_{n+1}[P']$

hence we get: $(C_n[Q],\ \mathtt{in}^{e'}_{\tilde{c}\tilde{x}\tilde{g}}(S(a\tilde{x}))) \twoheadrightarrow R = C_{n+1}[P']$, which implies $e, d, e' \notin \mathcal{AN}(R)$. Then, by (11), we have: $R = C_{n+1}[Q']$ with $Q \overset{\uparrow a(e_{n+1})}{\Longrightarrow}_a Q'$, as required.

Case $P \xrightarrow{\uparrow e_i(b)}_a P'$. With $S(a\tilde{x})$ as in the previous case, we have:

$(C_n[P], \mathtt{in}^{e'}_{\tilde{c}\tilde{x}\tilde{g}}(S(d_i\tilde{x}))) \twoheadrightarrow C_n[b\blacktriangleright(P, \leftarrow e_i b), S(d_i\tilde{e})] \rightarrow C_n[b\blacktriangleright(P, \leftarrow d_i b), S(d_i\tilde{e})] \rightarrow P''$

with $P'' = C_{n+1}[P']$ and the rest is just similar to the above using (12).

Exhausting all cases, we have shown that $\mathcal{S}$ is an asynchronous bisimulation. □

APPENDIX D. PROOF OF PROPOSITION 4.17

First, we easily show that there exists a synchronous weak bisimulation $\mathcal{S}$ such that $|\Im^*_{\pi=}| \subset \mathcal{S}$. The relation $\mathcal{S}$ is similarly defined as:

$$\mathcal{S} \overset{\text{def}}{=} \{\langle P,\ Q\rangle \mid \Im^*_{\pi=} \vdash C_n[P] = C_n[Q]\}$$

with $C_n[\ ]$ is the same context in Appendix C. Then the proof proceeds by checking each transition, using the exactly same contexts except in the case of the input transition. Suppose $P \xrightarrow{\downarrow ab}_\pi P'$. If $a \notin \{\tilde{e}\}$ and $b \notin \{\tilde{e}\}$, then we use the same context in the case of $\mathcal{BN}(l) = \emptyset$ as in the proof of Proposition 4.17 in the main paper. Other cases are also easy by replacing $\mathtt{in}^{e}_{\tilde{c}\tilde{g}\tilde{x}}(\leftarrow uv)$ in the case of the input transition in the proof in Appendix C with

$$\mathtt{in}^{e}_{\tilde{c}\tilde{g}\tilde{x}}(\bar{u}v.\bar{c}.\Lambda,\ c.\Lambda) \qquad (6)$$

where c is fresh. □

Appendix E. Proof of Proposition 5.5 (ii)

We show there exists $\mathcal{S}$ such that $\mathcal{S} \supseteq |\Im^{*}_{\nu:s}|$ and $\mathcal{S}$ is an asynchronous strong bisimulation. We define $\mathcal{S}$ as:

$$\mathcal{S} \stackrel{\text{def}}{=} \{\langle P,\ Q\rangle \mid \Im^{*}_{\nu:s} \vdash C_n[P] \ = \ C_n[Q]\}$$

where $C_n[\,] \stackrel{\text{def}}{=} \tilde{e} \blacktriangleright([\,],\ \prod_i^n !\mathsf{FW}(e_i d_i),\ \prod_i^n !\mathsf{SW}(c_i e_i))$ for some n with:

- names in $\tilde{e} \stackrel{\text{def}}{=} e_1 ... e_n$ pairwise distinct.
- names in $\tilde{c} \stackrel{\text{def}}{=} c_1 ... c_n$, $\tilde{d} \stackrel{\text{def}}{=} d_1 ... d_n$ fresh and pairwise distinct.

Note that $|\Im^{*}_{\nu:s}| \subset \mathcal{S}$ is obvious. We also use the following auxiliary expressions.

(i) $\mathtt{svc}(bcd) \stackrel{\text{def}}{=} by.(\leftarrow y,\ \leftarrow d),\ c.\Lambda,\ d.\Lambda$ ("svc" means a value checker in strong theories).

(ii) $\mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(R)$, as defined in Appendix C.

(iii) $S(ad_{n+1}c_{n+1}dlc) \stackrel{\text{def}}{=}$
$$ax.(!\mathsf{FW}(xd_{n+1}),\ !\mathsf{SW}(c_{n+1}x),\ \leftarrow xc,\ x.\leftarrow l,\ \leftarrow d), l.\Lambda,\ d.\Lambda$$

Below, as in Appendix C, we assume w.l.o.g. that whenever there is a transition or reduction from $C_n[P]$ reaching a certain term, the reduction/transition does not alter names in $\tilde{e}$ by α-conversion.

Notation: In the following, we write $P = Q$ for $\Im^{*}_{\nu:s} \vdash P = Q$.

Claim Let $C_n[\,]$ be the context as defined above, with names as specified there. Let a and b be names not included in $\{\tilde{d}\}$ (which cannot be in $\tilde{e}$ or $\tilde{c}$ either). For (8) to (11), we suppose the condition on names as stipulated in (7) of Claim in Appendix C.

(1) If $C_n[P] \to R$ and $\neg R \overset{\upharpoonright d_i}{\leadsto}$ (for any d_i), then $\exists P'.\ (R \equiv C_n[P'] \wedge P \to P')$.

(2) If $C_n[P] \xrightarrow{\upharpoonright ab}_a R$, then $\exists P'.\ (R \equiv C_n[P'] \wedge P \xrightarrow{\upharpoonright ab}_a P')$.

(3) If $C_n[P] \to\xrightarrow{\upharpoonright d_i b}_a R$, then $\exists P'.\ (R \equiv C_n[P'] \wedge P \xrightarrow{\upharpoonright e_i b}_a P')$.

(4) If, with d fresh and there is a unique occurrence of c in P, $(P, \mathtt{svc}(bcd)) \to^3 P'$ with c and d not occurring in P', we have $P \xrightarrow{\upharpoonright bc}_a P'$.

(5) If, with c and d fresh, $(C_n[P], \mathsf{SW}(ac), \mathtt{svc}(d_j cd)) \to^5 R$ with c and d not occurring in R, then $R \equiv C_n[P']$ with $P \xrightarrow{\upharpoonright a e_j}_a P'$.

(6) If, with c and d fresh, $(C_n[P], \mathsf{SW}(d_i c), \mathtt{svc}(d_j cd)) \to^6 R$ with c and d not occurring in R, then $R \equiv C_n[P']$ with $P \xrightarrow{\upharpoonright e_i e_j}_a P'$.

(7) If $(C_n[P], \leftarrow ab) \to R$, then $\exists P'.\ (R \equiv C_n[P'] \wedge P \xrightarrow{\downharpoonright ab}_a P')$.

(8) Suppose $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(R)) \rightarrow^{2n+1} R'$ and $e \notin \mathcal{AN}(R')$. Then $R' \equiv C_n[P, R\{\tilde{e}/\tilde{x}\}]$.

(9) If $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(\leftarrow x_i b)) \rightarrow^{2n+1} R$ and $e \notin \mathcal{AN}(R)$, then $\exists P'. \, (R \equiv C_n[P'] \wedge P \xrightarrow{\downarrow e_i b}_a P')$.

(10) If $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(\leftarrow a x_i)) \rightarrow^{2n+1} R$ and $e \notin \mathcal{AN}(R)$, then $\exists P'. \, (R \equiv C_n[P'] \wedge P \xrightarrow{\downarrow a e_i}_a P')$.

(11) If $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(\leftarrow x_i x_j)) \rightarrow^{2n+1} R$ and $e \notin \mathcal{AN}(R)$, then $\exists P'. \, (R \equiv C_n[P'] \wedge P \xrightarrow{\downarrow e_i e_j}_a P')$.

(12) Let $d_{n+1}, c_{n+1}, d, l, c$, fresh. Then, writing $S(a)$ for $S(a d_{n+1} c_{n+1} d l c)$, if $C_n[P], S(a) \rightarrow R_0 \rightarrow^3 R$ where $\neg R_0 \xrightarrow{\uparrow bc}_a$ for any b, and $l, d \notin \mathcal{AN}(R)$, then we have, with $C_{n+1}[\;] \overset{\text{def}}{=} \tilde{e} e_{n+1} \blacktriangleright([\;], \prod_i^{n+1} !\mathsf{FW}(e_i d_i), \prod_i^{n+1} !\mathsf{SW}(c_i e_i))$ where e_{n+1}, d_{n+1} and c_{n+1} fresh, we have $P \xrightarrow{\uparrow a(e_{n+1})}_a P'$ with $R \equiv C_{n+1}[P']$.

(13) Writing just $S(d_i)$ for $S(d_i d_{n+1} c_{n+1} d l c)$, if we have: $(C_n[P], S(d_i)) \rightarrow^2 R_0 \rightarrow^3 R$ where $\neg R_0 \xrightarrow{\uparrow bc}_a$ for any b, and $l, d \notin \mathcal{AN}(R)$, then, with the same $C_{n+1}[\;]$ as in (12), we have $P \xrightarrow{\uparrow e_i(b)}_a P'$ with $R \equiv C_{n+1}[P']$.

Below "(i)" denotes the first part of this proposition. (1), (2), .. are clauses in the Claim.

PROOF OF CLAIM: Mechanical from the definition of the context and Proposition 5.3 (ii). (1)–(3) are obvious. (4) is easy with a similar reasoning as in the proof of Proposition 5.5 (i). (5) and (6) follow from (4). (7) is trivial. (8) is because the reduction from $(C_n[P], \mathtt{in}^{e}_{\tilde{c}\tilde{x}\tilde{g}}(R))$ to $C_n[P, R\{\tilde{e}/\tilde{x}\}]$ should be determinate to eliminate $e.\Lambda$ in $2n+1$ steps. (9)–(11) are straightforward from (8). Regarding (12), since there should be interaction between $S(a)$ and $C_n[P]$ to eliminate d and l at the final step, it should be either:

(a) $P \xrightarrow{\uparrow ab}_a P'$ with:
$R_0 \equiv C_n[P', !\mathsf{FW}(b d_{n+1}), \; !\mathsf{SW}(c_{n+1} b), \; \leftarrow bc, \; x. \leftarrow l, \; \leftarrow d, l.\Lambda, \; d.\Lambda]$.

(b) $P \xrightarrow{\uparrow a(e_{n+1})}_a P'$, with: e_{n+1} fresh and $R_0 \equiv$
$C_n[e_{n+1} \blacktriangleright (P', !\mathsf{FW}(e_{n+1} d_{n+1}), \; !\mathsf{SW}(c_{n+1} e_{n+1}), \; \leftarrow e_{n+1} c, \; x. \leftarrow l, \; \leftarrow d), l.\Lambda, \; d.\Lambda]$.

But if (a) holds, then $R_0 \xrightarrow{\uparrow bc}_a$, contradicting the assumption. Therefore (b) holds, then $R_0 \rightarrow^3 C_{n+1}[P']$ is immediate, hence (12) holds. Finally (13) uses (3) and (12).
□

Now assume $\langle P,\ Q\rangle \in \mathcal{S}$. We show that $P \xrightarrow{l}_a P'$ implies for some Q', $Q \xrightarrow{l}_a Q'$ with $\langle P',\ Q'\rangle \in \mathcal{S}$ by checking each transition. Below, as before, "(i)" denotes the first part of this proposition. (1), (2), .. are clauses in the Claim.

Case $P \xrightarrow{\tau}_a P'$: Note that $\mathcal{FN}(P') \cap \{\tilde{d}\} = \emptyset$. Then:

$$\begin{array}{lcll} C_n[P] \to C_n[P'] & \Rightarrow & C_n[Q] \to R = C_n[P'] & \text{(reduction-closure)} \\ & \Rightarrow & R \equiv C_n[Q'] \wedge Q \ \to Q' & \text{(1) with } \mathcal{FN}(Q) \cap \{\tilde{d}\} = \emptyset \end{array}$$

Case $P \xrightarrow{\uparrow ab}_a P'$. Then:

$$\begin{array}{lcll} C_n[P] \xrightarrow{\uparrow ab}_a C_n[P'] & \Rightarrow & C_n[Q] \xrightarrow{\uparrow ab}_a R = C_n[P'] & \text{(i)} \\ & \Rightarrow & R \equiv C_n[Q'] \wedge Q \xrightarrow{\uparrow ab}_a Q' & \text{(2)} \end{array}$$

Case $P \xrightarrow{\uparrow e_i b}_a P'$. Then $C_n[P] \to\xrightarrow{\uparrow d_i b}_a C_n[P']$ and the rest is similar to the above, using (3) instead.

Case $P \xrightarrow{\uparrow a e_j}_a P'$. Then, with c, d fresh,

$$\begin{array}{lll} (C_n[P], \mathsf{SW}(ac), \mathtt{svc}(d_j cd)) \to^5 C_n[P'] & & \\ \quad \Rightarrow \quad (C_n[Q], \mathsf{SW}(ac), \mathtt{svc}(d_j cd)) \to^5 R = C_n[P'] & \text{(reduction closure)} \\ \quad \Rightarrow \quad R \equiv C_n[Q'] \ \wedge \ Q \xrightarrow{\uparrow a e_j}_a Q' & \text{(5)} \end{array}$$

Case $P \xrightarrow{\uparrow e_i e_j}_a P'$. Then, with c, d fresh, $(C_n[P], \mathsf{SW}(d_i c), \mathtt{svc}(d_j cd)) \to^6 C_n[P']$ and the rest is similar to the above, using (6) instead.

Case $P \xrightarrow{\downarrow ab}_a P'$. Trivial by (7).

Case $P \xrightarrow{\downarrow e_i b}_a P'$. Easy by (9).

Case $P \xrightarrow{\downarrow a e_j}_a P'$. Also easy by (10).

Case $P \xrightarrow{\downarrow e_i e_j}_a P'$. Also easy by (11).

Case $P \xrightarrow{\uparrow a(e_{n+1})}_a P'$. Then we have:

$$C_n[P], S(a) \to (C_{n+1}[P',\ \leftarrow e_{n+1} c, e_{n+1}. \leftarrow l],\ \leftarrow d,\ d.\Lambda,\ l.\Lambda) \stackrel{\text{def}}{=} P_0 \ \to^3 \ C_{n+1}[P']$$

which requires $(C_n[Q], S(a)) \to Q_0 = P_0$ and $Q_0 \to^3 Q'' = C_{n+1}[P']$. Then, by Q'' being equal to $C_{n+1}[P']$, $\mathcal{AN}(Q'')$ cannot contain d or l, hence $C_n[P]$ and $S(a)$ interacts with each other at the first step (at a). But by $\neg P_0 \xrightarrow{\uparrow bc}_a$ for any b, we know $\neg Q_0 \xrightarrow{\uparrow bc}_a$ for any b by (i), thus the assumption of (12) is satisfied, hence $Q'' \equiv C_{n+1}[Q']$ (i.e. $P' \,\mathcal{S}\, Q'$) with $Q \xrightarrow{\uparrow a(e_{n+1})}_a Q'$. Thus this case holds.

Case $P \xrightarrow{\uparrow e_i(e_{n+1})}_a P'$. The result is obtained by composing "$S(d_i)$" instead of "$S(a)$" of the preceding case, together with (13).

Hence $\mathcal{S}$ is a strong asynchronous bisimulation, concluding the proof. □