Açores, 31 Août 2000

LINEAR
LOGIC
AN
INTRODUCTION

Jean-Yves Girard

Hilbert's Program

1900 : Second problem (consistency).

1902 : This is a mathematical problem.

1925 : On the infinite : emphasis on Π^0_1 (i.e. Σ^1, recessive) properties, consistency, non-termination… ; elimination of infinity.

1931 : Gödel's theorem : failure of the program. True unprovable Σ^1 formulas.

Constructivity

- Poincaré, Brouwer, intuitionism. Success in the 1920s.
- A proof has a implicit contents. Disjunction property « A proof of $A \vee B$ » is implicitly either a proof of A or a proof of B. Refusal of tertium non datur $A \vee \neg A$ and contraposition $\neg\neg A = A$.
- Existence property (for numerical quantifiers $\exists n A$).
- Ended in sectarian fights (constructivism)

Gentzen

- 1934 : Sequent calculus, cut-elimination : elimination of the only intelligent rule, i.e. Modus Ponens.
- Disjunction property, due the shape of intuitionistic sequents $\Gamma \vdash A$. Last cut-free rule to prove $\vdash A \vee B$ is a disjunction rule.
- Subformula property for first-order formulas, more generally Π^1 formula.
- Compare with A provable iff A true when A is Π^1. Internal form of completeness.
- Further work of Gentzen : consistency of arithmetic, ended in Panzerdivisionen.

Lambda-calculus

- Church, Rosser, Kleene.
- Naive theory of functions, compare $a \in b$ with $f(g)$, $\{x; P\}$ with $\lambda x T$.
- Russell's paradox applies ; $\{x; x \notin x\}$ becomes $\lambda x M(x(x))$.

$$\Omega\Omega = M(\Omega\Omega)$$

- In λ-calculus the contradiction is solved by divergence of the computation... a stupid idea that eventually works.

Ludic traditions

- 1936 : First consistency proof of Gentzen. A proof P induces a sort of winning strategy in the game induced by A.
- Refused car of no fundational value : compare with If A is provable then A is true.
- Next proof by Gentzen used ϵ_0 . . .
- Dialectica (before 1958) interpretation of Gödel. Interpretation $\exists X \forall Y A[X, Y]$. X for the strategy, Y for the counterstrategy. Awkward and leaks a lot.
- 1960 : Lorenzen, sort of dialectics. . . completely ad hoc, the Frankenstein monster.

The roaring sixties

- Influence of Kreisel, change of viewpoint, forget isms.
- 1958 : HRO, HEO partial equivalence relations.
- Prawitz : Natural Deduction 1965, symmetry introduction/elimination.
- Dana Scott 1969 : a CCC of topological spaces. Topology $\mathcal{T}_0$, the weakest form of separation... not quite a topology, but still first mathematical explanation of λ-calculus.

The time of categories 1970-2000

- Curry-Howard 1969 : proofs as functions.
- System $\mathbb{F}$, Girard 1970,
  - Extension of Dialectica to second order.
  - Extension of Curry-Howard
  - Proof of Takeuti's conjecture : second-order cut-elimination.
- Dilators, Girard 1976 : preservation of pull-backs

$$D(f \mathbin{\&} g) = D(f) \mathbin{\&} D(g)$$

- Denotational semantics : Berry 1978, discovery of stability

$f(a \cap b) = f(a) \cap f(b)$ if $a \cup b$ exists.

New interpretations

Quantitative semantics, 1983 : Functions as power series

$$\sum \alpha_I c^I$$

Qualitative semantics 1984 : Simplified Scott semantics, makes use of preservation of pull-backs/stability.

Discovery of linear logic, 1985

- Linearity : $\sum \alpha_{\{i\}} c^{\{i\}}$.
- Use of « symmetric tensor algebra » to recover full case.
- Founding equation :

$$A \Rightarrow B = !A \multimap B$$

- Stability = feedback : $x \in F(a)$ comes from well-defined finite $b \subset a$: gives rise to linear negation, the most important operation.

$$A \multimap B = B^{\perp} \multimap A^{\perp}$$

- Qualitative domains become binary : coherent spaces.

The meaning of linearity

- $F(x) = a$ weakening, constant function.
- $F(x) = G(x, x)$ contraction, quadratic function.
- Intuitionistic disjunction property : no contraction can be performed on the right. Asymetry left/right.
- Linear logic : refuse weakening and contraction ; reintroduce as logical rules. Symmetry left/right restored. . . possibility of ludic interpretations.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ (\textit{weak.}) \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ (\textit{cont.})$$

The new connectives

- Splitting of conjunction between

Multiplicatives : Tensor $\otimes$, Par $\parr$

Additives : Plus $\oplus$, With $\&$

Exponentials : Of course $!$, Why not $?$

Fundamental equation : Relates the two conjunctions.

$$!(A \mathbin{\&} B) = !A \otimes !B$$

Compare with

$$A \wedge B \Rightarrow C = (A \Rightarrow (B \Rightarrow C))$$

Proof-nets

- The connective $\otimes$ is positive (works like intuitionistic disjunction). Problems with natural deduction (think of negation too).
- Proof-nets 1986 : non-sequential syntax, global correctness criterion.
- Interpretation in terms of permutations over a finite set :

  $$\sigma \perp \tau \text{ iff } \sigma\tau \text{ cyclic.}$$
- Geometry of interaction, 1988 : replaces permutations by operators.

Completeness issues

- Restricted to Π^1 formulas (Gödel incompleteness).
- One-dimensional version : phase space ; decision problems.
- Two-dimensional version : denotational semantics. Leaks, since usually difficult to exclude *bad* objects. But interesting as a treason.
- Three-dimensional version : w.r.t. ludic (dynamic) interpretation. Works for small fragments, e.g. MLL_2 without neutrals. Tendency to leaking, e.g. adjunction of dubious principles, typically the mix rule

$$A, B \vdash A, B$$

- Main problem : get rid of the referree ; no need for him.

Ludics

- The key leading to ludics is focalisation, Andreoli 1989.
  $A \mathbin{\&} (B \parr C)$ is a ternary connective, but not $A \mathbin{\&} (B \oplus C)$.
- Logical time : alternation of polarities.
- Interactive approach without referree, games without rule.
  Designs behaviours . . .
- Full completeness for $MALL_2$. Most difficult problem : to handle second-order quantifiers from the inside.
- If $\mathfrak{D} \in \mathbf{A}$ is winning i.e. uniform, stubborn and parsimonious, then there is a proof π of A such that $\boldsymbol{\pi} = |\mathfrak{D}|$.
- The actual logical space : the losers.

Açores, 3 Septembre 2000

FROM
PROOFS
TO
DESIGNS

Jean-Yves Girard

Pierre Ménard

El método inicial que imaginó era relativamente sencillo. Conocer bien el español, recuperar la fe católica, guerrear contra los moros o contra el turco, olvidar la historia de Europa entre los años de 1602 y de 1918, ser Miguel de Cervantes. Pierre Ménard estudió ese procedimiento (sé que logró un manejo bastante fiel del español del siglo diecisiete), pero lo descartó por fácil. [. . .]

Mi complaciente precursor no rehusó la colaboración del azar : iba componiendo la obra inmortal un poco à la diable, llevado por inercias del lenguaje y de la invención. Yo he contraído el misterioso deber de reconstruir literalmente su obra espontánea. Mi solitario juego está gobernado por dos leyes polares. La primera me permite ensayar variantes de tipo formal o psicológico ; la segunda me obliga a sacrificarlas al texto « original » y a razonar de un modo irrefutable esa aniquilación.

J. L. Borges : Pierre Ménard autor del Quijote, 1939.

Analysis : time in logic

- Two polarities :

  Negative : invertible logical connectives[a]. $(\wedge, \Rightarrow), \&, \parr, \top, \bot, \forall$.

  Positive : you commit yourself, but can perform clusters (Andreoli).
  $(\vee), \oplus, \otimes, 0, 1, \exists$. Association of connectives of same polarity[b].

- Clock incremented by alternation of polarities[c].

- Paradigm extended to atoms : X is positive, no identity axiom, rather infinite non-well founded η-expansion.

- Restriction to sequents $\vdash \Gamma$ with at most one negative formula. Rewrite as $\vdash \Gamma$ (positive sequent) or $A \vdash \Gamma$ (negative sequent).

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[a] Going « downwards » in natural deduction.

[b] Graphical styles mnemonise association, e.g. distribution $\otimes/\oplus$.

[c] $\Phi(A, B, C) = A \,\&\, (B \oplus C)$ is not a connective.

Analysis : space in logic

- Forget formulas, remember only locations : locus solum.
  - Loci are addresses of subformulas $\xi = \langle i_1, \ldots, i_n \rangle$
  - $i_k \in \mathbb{N}$ is called a bias.
  - Example : P, Q, R, immediate subformulas of A (located at ξ) are located at addresses correspond to $\xi * 3, \xi * 4, \xi * 7$
  - Loci are either disjoint (space) or comparable (time).

Analysis : sequent calculus

- Replace connectives with clusters, and combine with negation. See $((L \oplus M) \otimes N)$ as $\Phi(L^\perp, M^\perp, N^\perp)$.
- Write cluster rules

$$\frac{P \vdash \Gamma \quad R \vdash \Delta}{\vdash \Gamma, \Delta, \Phi(P^\perp, Q^\perp, R^\perp)}\,(l\vdash \Phi)$$

$$\frac{\vdash \Lambda, P, R \quad \vdash \Lambda, Q, R}{\Phi(P^\perp, Q^\perp, R^\perp) \vdash \Lambda}\,(\Phi \vdash)$$

$$\frac{Q \vdash \Gamma \quad R \vdash \Delta}{\vdash \Gamma, \Delta, \Phi(P^\perp, Q^\perp, R^\perp)}\,(r\vdash \Phi)$$

Designs : example of rules

- Sequents become pitchforks $\Xi \vdash \Lambda$
  - Ξ, Λ finite sets of addresses, pairwise disjoint (handle, tines).
  - $\sharp(\Xi) \leq 1$; $\vdash \Lambda$ is a brush —positive— ; $\xi \vdash \Lambda$ is negative.
  - Previous rules rewrite as :

$$\dfrac{\xi3 \vdash \Gamma \qquad \xi7 \vdash \Delta}{\vdash \Gamma, \Delta, \xi}\,{\scriptstyle(\vdash\xi,\{3,7\})}$$

$$\dfrac{\vdash \Lambda, \xi3, \xi7 \qquad \vdash \Lambda, \xi4, \xi7}{\xi \vdash \Lambda}\,{\scriptstyle(\xi\vdash\{\{3,7\},\{4,7\}\})}$$

$$\dfrac{\xi4 \vdash \Gamma \qquad \xi7 \vdash \Delta}{\vdash \Gamma, \Delta, \xi}\,{\scriptstyle(\vdash\xi,\{4,7\})}$$

Ludics : Designs

Demon : too late !

$$\frac{}{\vdash \Lambda}\ \pitchfork$$

Positive rule : [a]I is a ramification, for $i \in I$ the Λ_i are pairwise disjoint and included in Λ : one can apply the rule (finite, one premise for each $i \in I$)

$$\frac{\ldots, \xi * i \vdash \Lambda_i, \ldots}{\vdash \Lambda, \xi}\ (\xi, I)$$

Negative rule : [b]$\mathcal{N}$ is a set of ramifications, for all $I \in \mathcal{N}$ $\Lambda_I \subset \Lambda$: one can apply the rule : perhaps infinite, one premise for each $I \in \mathcal{N}$)

$$\frac{\ldots, \vdash \Lambda_I, \xi * I, \ldots}{\xi \vdash \Lambda}\ (\xi, \mathcal{N})$$

- No assumption of finiteness, well-foundedness, recursivity etc.

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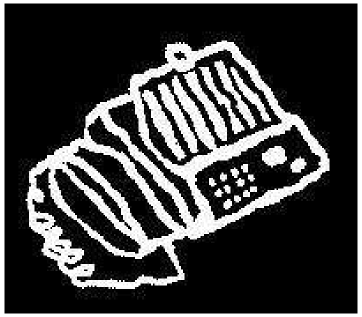
[a] Abstract form of $\oplus \otimes (.)^{\perp}$.

[b] Abstract form of $\& \parr (.)^{\perp}$.

The fax



- If ξ and ξ' are disjoint, then one defines $\mathfrak{Fax}_{\xi,\xi'}$, a design of basis $\xi \vdash \xi'$.

$$\dfrac{\ldots \qquad \dfrac{\begin{array}{c}\vdots\ \mathfrak{Fax}_{\xi' * i, \xi * i} \\ \ldots \xi' * i \vdash \xi * i \ldots\end{array}}{\vdash \xi', \xi * I}\,(\xi', I) \qquad \ldots}{\xi \vdash \xi'}\,(\xi, \wp_f(\mathbb{N}))$$

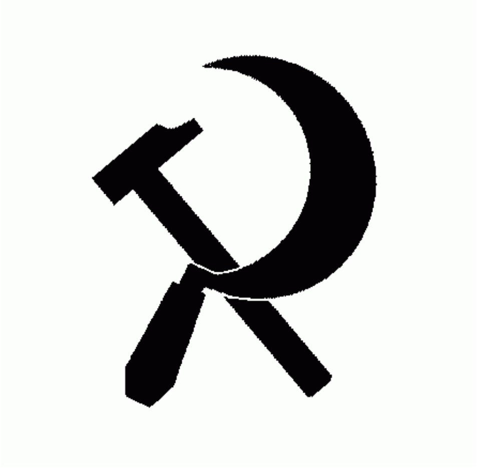
The Daimon

Açores, 3 Septembre 2000

Daimon, 𝔇𝔞𝔦 :

$$\dfrac{}{\vdash \Lambda}\ \pitchfork$$

Faith



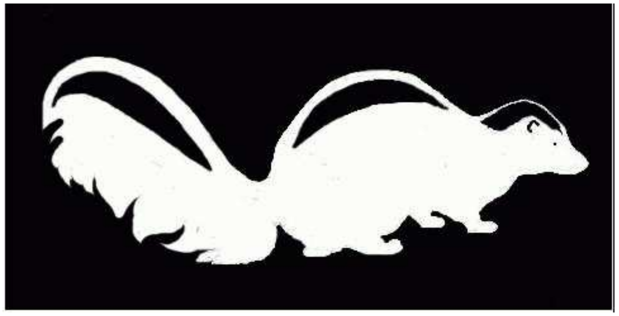
Faith, $\mathfrak{Fid}$:

$$\frac{}{\vdash \Lambda}\ \Omega$$

notation $\mathfrak{Fid}$. Not a design, the only *partial* design. The exact analogue of the non-solvable term. Think of designs as sort of symmetric Böhm trees.

On basis $\vdash$ only two possibilities, $\mathfrak{Dai}$ (converges), $\mathfrak{Fid}$ (diverges).

The Skunk.



Skunk, $\mathfrak{Sk}$:

$$\frac{}{\xi \vdash \Lambda}\ (\xi, \emptyset)$$

The smallest negative design ; very asocial, notation $\mathfrak{Sk}$.

The negative daimon.

Negative daimon, $\mathfrak{Dai}^-$:

$$\dfrac{\ldots \qquad \dfrac{}{\vdash \xi * I, \Lambda}\,\pitchfork \qquad \ldots}{\xi \vdash \Lambda}\,(\xi, \wp_f(\mathbb{N}))$$

Analysis : cut-elimination

- The rule $\Phi\vdash$ has 2 premises, one for each rule $\vdash\Phi$
- The right premise of $\Phi\vdash$ matches the right rule $\vdash\Phi$.
- Cut between $\Phi(P^\perp, Q^\perp, R^\perp)\vdash\Lambda$ and $\vdash\Gamma,\Delta,\Phi(P^\perp, Q^\perp, R^\perp)$ replaced with two cuts between $\vdash\Lambda, Q, R,\quad Q\vdash\Gamma,\quad R\vdash\Delta$[a].

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[a] Order of cuts irrelevant : use cut-links, and not cut-rules !

Nets

- A cut is a coincidence handle/tine between two bases of designs. More generally define nets : incidence graph connected/acyclic. A net has a base, the non-shared locii. Closed net when basis is $\vdash$. Example $\{\xi \vdash ; \xi \vdash \lambda\}$, one cut on ξ, base $\vdash \lambda$.
- Example : negative rule corresponds to $\mathcal{N} = \{\{3,7\},\{4,7\}\}$, positive rules correspond to $\{3,7\}$ or $\{4,7\}$. Matching $I \in \mathcal{N}$ otherwise divergence.
- Cut with Daimon always converges : types are non-empty.

Açores, 4 Septembre 2000

FROM
DESIGNS
TO
BEHAVIOURS

Jean-Yves Girard

Normalisation

Cut-nets : Cut defined by coincidence handle/tine between two designs. $\{\mathfrak{D}_0, \ldots, \mathfrak{D}_n\}$ induces a connected and acyclic graph. Example : $\{\xi \vdash \xi'; \xi' \vdash\}$.

Normalisation : $[\![\mathfrak{D}_0, \ldots, \mathfrak{D}_n]\!]$ is the normal form of $\{\mathfrak{D}_0, \ldots, \mathfrak{D}_n\}$. In case of divergence (positive base only) use notation $\mathfrak{Fid}$.

Example : the fax

- $\mathfrak{D}, \mathfrak{E}$ of bases $\vdash \langle\rangle$ and $\langle\rangle \vdash$.
- Φ, Ψ delocations : $\Phi(\sigma) = \xi * \sigma$, $\Psi(\sigma) = \xi' * \sigma$.
- $[\![\mathfrak{Fax}, \Phi(\mathfrak{D})]\!] = \Psi(\mathfrak{D})$
- $[\![\mathfrak{Fax}, \Psi(\mathfrak{E})]\!] = \Phi(\mathfrak{E})$
- $[\![\mathfrak{Fax}, \Phi(\mathfrak{D}), \Psi(\mathfrak{E})]\!] = [\![\mathfrak{D}, \mathfrak{E}]\!]$
- Eventually $\mathfrak{Fax}_{\xi,\xi'}$ will inhabit

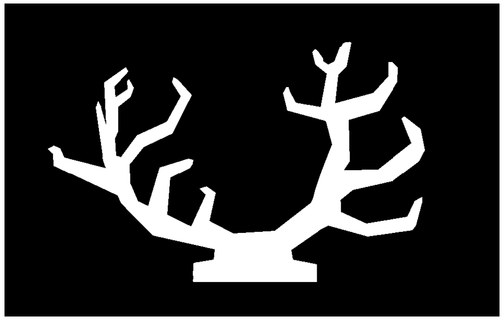
$$\bigcap \mathbf{X}[\Phi(\mathbf{X}) \vdash \psi(\mathbf{X})]$$

.

Example : daimons

- $[\![\mathfrak{Dai}, \mathfrak{D}_1, \ldots, \mathfrak{D}_n]\!] = \mathfrak{Dai}$.
  Positive daimon definitely very social.
- Closed case, $\vdash \xi$ vs. $\xi \vdash$:
  - $[\![\mathfrak{D}, \mathfrak{Dai}^-]\!] = \mathfrak{Dai}$.
  - $[\![\mathfrak{D}, \mathfrak{Sk}]\!] = \mathfrak{Fid}$ (diverges) but when $\mathfrak{D} = \mathfrak{Dai}$.
  - More generally, $[\![\mathfrak{D}, \mathfrak{Dir}_{\mathcal{N}}]\!]$ diverges exactly when $\mathfrak{D}$ has first rule $\vdash (\xi, I)$ with $I \notin \mathcal{N}$.

Example : directory



Directory, $\mathfrak{Dir}_{\mathcal{N}}$:

$$\dfrac{\ldots \qquad \dfrac{}{\vdash \xi * I, \Lambda}\pitchfork \qquad \ldots}{\xi \vdash \Lambda}\,(\xi,\mathcal{N})$$

$\mathfrak{Sk} = \mathfrak{Dir}_{\emptyset}$

$\mathfrak{Dai}^{-} = \mathfrak{Dir}_{\wp_f(\mathbb{N})}$

Example : boots

Boots, $\mathfrak{Boots}$:

$$\dfrac{\dfrac{}{\vdash}\ \pitchfork}{\xi \vdash}\ (\xi,\{\emptyset\})$$

$\mathfrak{Boots} = \mathfrak{Dir}_{\{\emptyset\}}$

Separation

Orthogonality : If $\mathfrak{D}, \mathfrak{E}$ of bases $\vdash \xi$ and $\xi \vdash$,
$\mathfrak{D} \perp \mathfrak{E}$ means that $[\![\mathfrak{D}, \mathfrak{E}]\!]$ converges, i.e. $[\![\mathfrak{D}, \mathfrak{E}]\!] = \mathfrak{Dai}$.

Separation : Topology generated by sets $\mathfrak{E}^{\perp}$ is $\mathcal{T}_0$. In other terms $\mathfrak{D}$ is determined by its orthogonal.

Dessins vs Desseins : In fact true if one gets rid of irrelevant contexts.

Ordering of designs : $\mathfrak{D} \preceq \mathfrak{E} =: \mathfrak{D}^{\perp} \subset \mathfrak{E}^{\perp}$.

$$\mathfrak{Fid} \preceq (\xi, I) \preceq \mathfrak{Dai}$$

Böhm theorem : Designs as Böhm trees, separation as Böhm theorem, $\preceq$ as extensional order.

Associativity

Church-Rosser : $[\![[\![\mathfrak{R}_1]\!], \ldots, [\![\mathfrak{R}_n]\!]]\!] = [\![\mathfrak{R}_1, \ldots, \mathfrak{R}_n]\!]$.

Example : $[\![\mathfrak{Fax}, \Phi(\mathfrak{D}), \Psi(\mathfrak{E})]\!] = [\![[\![\mathfrak{Fax}, \Phi(\mathfrak{D})]\!], \Psi(\mathfrak{E})]\!]$.

Closure principle : Equation $[\![\mathfrak{Fax}, \Phi(\mathfrak{D})]\!] = \Psi(\mathfrak{D})$ characterises the fax.
Combination of associativity and separation.

Adjunctions : In order to prove associativity of $⅋$, one later needs the equivalence between the two definitions

$$\mathfrak{F} \in \mathbf{G} ⅋ \mathbf{H} \Leftrightarrow \forall \mathfrak{A}(\mathfrak{A} \in \mathbf{G}^{\perp} \Rightarrow [\mathfrak{F}]\mathfrak{A} \in \mathbf{H})$$

$$\mathfrak{F} \in \mathbf{G} ⅋ \mathbf{H} \Leftrightarrow \forall \mathfrak{B}(\mathfrak{B} \in \mathbf{H}^{\perp} \Rightarrow [\mathfrak{F}]\mathfrak{B} \in \mathbf{G})$$

This is basically the closure principle.

Stability

Stable order : Just plain inclusion, coarser than $\preceq$.

Stability : Normalisation commutes to arbitrary intersections.

$$[\![\mathfrak{R}_1 \cap \mathfrak{R}_2]\!] = [\![\mathfrak{R}_1]\!] \cap [\![\mathfrak{R}_2]\!]$$

Nothing like existence of union is needed ; however relies on existence of intersection as a dessein.

Monotonicity : $\mathfrak{R} \preceq \mathfrak{S} \Rightarrow [\![\mathfrak{R}]\!] \preceq [\![\mathfrak{S}]\!]$

Atomic weapons

One, $\mathfrak{One}$: No way to answer, wins anyway !

$$\frac{}{\vdash \langle\rangle}\ (\langle\rangle,\emptyset)$$

A rule of game : Can we forbid atomic weapons ?

$[\![\mathfrak{D}, \mathfrak{Dir}_{\wp_f(\mathbb{N})-\emptyset}]\!]$ diverges exactly when $\mathfrak{D} = \mathfrak{One}$.

Behaviours : The rule of game is interactive too ! Guys in charge of rule are losers… think of a dog, an independent prosecutor…

Behaviours

Definition : A behaviour (of given base $\vdash \xi$ or $\xi \vdash$) $\mathbf{G}$ is a set of designs equal to its biorthogonal.

Non empty : $\mathfrak{Dai}^{\epsilon} \in \mathbf{G}$ if $\mathbf{G}$ is of polarity ϵ.

Closure : $\mathfrak{D} \in \mathbf{G}$ and $\mathfrak{D} \preceq \mathfrak{E} \Rightarrow \mathfrak{E} \in \mathbf{G}$.

Stability : $\mathbf{G}$ closed under intersection... provided it exists ?

Rule : Most of counterdesigns are in charge of the rule of the game. Idea of consensus, think of World War I.

Encarnación

Equivalence : $\mathfrak{D} \in \mathbf{G}$ and $\mathfrak{D} \subset \mathfrak{E} \Rightarrow \mathfrak{E} \in \mathbf{G}$; say that $\mathfrak{D} \sim_{\mathbf{G}} \mathfrak{E}$. Can we avoid this equivalence ?

Incarnation : Let $|\mathfrak{D}|_{\mathbf{G}} = \bigcap\{\mathfrak{D}'; \mathfrak{D}' \subset \mathfrak{D} \text{ and } \mathfrak{D}' \in \mathbf{G}\}$. Then

$$\mathfrak{D} \sim \mathfrak{E} \Leftrightarrow |\mathfrak{D}|_{\mathbf{G}} = |\mathfrak{E}|_{\mathbf{G}}$$

Contravariance : $\mathfrak{D} \in \mathbf{G} \subset \mathbf{H} \Rightarrow |\mathfrak{D}|_{\mathbf{H}} \subset |\mathfrak{D}|_{\mathbf{G}}$.

Principal behaviour : $\mathfrak{D}$ has maximum incarnation $\mathfrak{D}$ in principal behaviours $\mathfrak{D}^{\perp\perp}$, the smallest behaviour containing $\mathfrak{D}$.

Skunk's social life : $\mathfrak{Sk}^{\perp\perp}$ consists of all (negative) designs. But the only incarnated design in this behaviour is $\mathfrak{Sk}$.

Claude Debussy

Oui, c’est imbécile ce que je dis ! Seulement je ne sais pas comment concilier tout ça. Il est sûr que je ne me sens libre que parce que j’ai fait mes classes et que je ne sors de la fugue que parce que je la sais.

Claude-Achille Debussy Entretiens avec Ernest Guiraud, ~ 1890.

Açores, 5 Septembre 2000

ADDITIVE AND MULTIPLICATIVE CONNECTIVES

Jean-Yves Girard

Delocations

- Injective map θ from the subloci of ξ to the subloci of ξ' such that
  - $\theta(\xi) = \xi'$
  - For all σ there is a function θ_σ such that $\theta(\sigma * i) = \theta(\sigma) * \theta_\sigma(i)$.
  - Image $\theta(\mathfrak{D})$ of a design of base $\vdash \xi$ is a design of base $\vdash \xi'$, etc.
  - The set of designs

    $$\mathbf{E} = \{\theta(\mathfrak{D}); \mathfrak{D} \in \mathbf{G}\}$$

    is not a behaviour, but only an ethics for $\theta(\mathbf{G}) = \mathbf{E}^{\perp\perp}$.
- Internal completeness : up to incarnation, $\theta(\mathbf{G}) = \mathbf{E}$, i.e. $|\theta(\mathbf{G})| \subset \mathbf{E}$.

Shifts

Anderssen opening : Dummy move `a2-a3`. For us, pure change of polarity, unary case of tensor/plus.

Positive shift : If $\mathfrak{D}$ of base $\xi * i \vdash$, define $\downarrow\mathfrak{D}$ of base $\vdash \xi$: add « first move », i.e. positive rule $(\xi, \{i\})$.

Negative shift : If $\mathfrak{D}$ of base $\vdash \xi * i$, define $\uparrow\mathfrak{D}$ of base $\xi \vdash$: add « first move », i.e. negative rule $(\xi, \{\{i\}\})$.

Internal completeness : The sets $\updownarrow\mathbf{G} = \{\updownarrow\mathfrak{D}; \mathfrak{D} \in \mathbf{G}\}$ are complete ethics.

Strict connectives

Strict connectives : Additives and multiplicatives enjoy equalities.

Associativity

Commutativity

Neutral

Absorber

Distributivity of multiplicatives over additives of same polarity.

Incompleteness : A union is usually not a complete ethics ; think of second-order existential quantification.

The locative dilemma

Sipiritual hypothesis : Behaviours are « disjoint » enough.

By accident : The atoms happen to be located in the right place.

By construction : Systematically use delocations

$$\varphi(i * \sigma) = 3i * \sigma$$

$$\psi(i * \sigma) = (3i + 1) * \sigma$$

and redefine $\oplus$ as $\varphi(\mathbf{G}) \oplus \psi(\mathbf{H})\ldots$

Twins : Distinct persons or distinct occurrences of the same person ?

Boots, One : No nice delocation : excluded from completeness (but shifts can be handled).

Example : ramification

Ramification, $\mathfrak{Ram}_I$:

$$\dfrac{\dots \qquad \dfrac{\dfrac{}{\dots \vdash \lambda * i * J}\,\pitchfork \qquad \dots}{\lambda * i \vdash}\,(\lambda * i, \wp_f(\mathbb{N})) \qquad \dots}{\vdash \lambda}\,(\lambda, I)$$

The directory of a behaviour

Positive case : $\P\mathbf{G} = \{I; \mathfrak{Ram}_I \in \mathbf{G}\}$.

Negative case : $|\mathfrak{Dai}^-|_{\mathbf{G}} = \mathfrak{Dir}_{\P\mathbf{G}}$.

Spiritual hypothesis : $\mathbf{G}, \mathbf{H}$ disjoint iff directories are disjoint

$$\P\mathbf{G} \cap \P\mathbf{H} = \emptyset$$

The disjunction property

- If $\mathbf{G}, \mathbf{H}$ are positive and disjoint, then

$$\mathbf{G} \oplus \mathbf{H} = \mathbf{G} \cup \mathbf{H}$$

- Internal completeness of $\oplus$.
- Unique decomposition into connected behaviours

$$\mathbf{G} = \bigoplus_{I \in \P\mathbf{G}} \mathbf{G}_I$$

The mystery of incarnation

- If $\mathbf{G}, \mathbf{H}$ are negative and disjoint, then

$$|\mathbf{G} \mathbin{\&} \mathbf{H}| = |\mathbf{G}| \times |\mathbf{H}|$$

- Use locative product :

$$X \uplus Y = \{x \cup y; x \in X, y \in Y\}$$

- Internal completeness of $\&$.
- Unique decomposition into connected behaviours

$$\mathbf{G} = \mathop{\Large\&}_{I \in \P\mathbf{G}} \mathbf{G}_I$$

The tensor product

- If $\mathfrak{D}, \mathfrak{E}$ are positive designs of base $\vdash \langle\rangle$, define tensor product :

  Unproper case : $\mathfrak{Dai}$ if one of $\mathfrak{D}, \mathfrak{E}$ is a daimon.

  Disjoint case : Disjoint first actions I, J, replace with $I \cup J$ and glue.

  Non disjoint case : Four protocols

  - $\mathfrak{D}$ has priority on $I \cap J$, notation $\mathfrak{D} \olessthan \mathfrak{E}$.
  - $\mathfrak{E}$ has priority on $I \cap J$, notation $\mathfrak{D} \ogreaterthan \mathfrak{E}$.
  - $\mathfrak{Dai}$, notation $\mathfrak{D} \odot \mathfrak{E}$.
  - Put skunks on $I \cap J$, notation $\mathfrak{D} \obar \mathfrak{E}$.

The adjunctions

- Each of the four tensors has an adjoint application :

Non commutative case :

$$\ll \mathfrak{F} \mid \mathfrak{A} \olessthan \mathfrak{B} \gg \, = \, \ll \mathfrak{F}[\mathfrak{A}] \mid \mathfrak{B} \gg \, = \, \ll \mathfrak{A} \mid [\mathfrak{B}]\mathfrak{F} \gg$$

Commutative cases :

$$\ll \mathfrak{F} \mid \mathfrak{A} \odot \mathfrak{B} \gg \, = \, \ll [\mathfrak{F}]\mathfrak{A} \mid \mathfrak{B} \gg \; = \; \ll \mathfrak{A} \mid [\mathfrak{F}]\mathfrak{B} \gg$$

$$\ll \mathfrak{F} \mid \mathfrak{A} \obar \mathfrak{B} \gg \, = \, \ll \{\mathfrak{F}\}\mathfrak{A} \mid \mathfrak{B} \gg \; = \; \ll \mathfrak{A} \mid \{\mathfrak{F}\}\mathfrak{B} \gg$$

The tensors

- The negative connectives $\bowtie$, $\ltimes$, $\rtimes$, ∞ defined by :

$$\mathfrak{F} \in \mathbf{G} \ltimes \mathbf{H} \Leftrightarrow \forall \mathfrak{A} (\mathfrak{A} \in \mathbf{G}^{\perp} \Rightarrow \mathfrak{F}[\mathfrak{A}] \in \mathbf{H})$$

are associative, distributive… neutral $\mathfrak{Boots}^{\perp\perp}$: equivalent definition

$$\mathfrak{F} \in \mathbf{G} \ltimes \mathbf{H} \Leftrightarrow \forall \mathfrak{B} (\mathfrak{B} \in \mathbf{H}^{\perp} \Rightarrow [\mathfrak{B}]\mathfrak{F} \in \mathbf{G})$$

- As a consequence, the four tensors

$$\mathbf{G} \circledast \mathbf{H} = \{\mathfrak{A} \circledast \mathfrak{B} ; \mathfrak{A} \in \mathbf{G}, \mathfrak{B} \in \mathbf{H}\}^{\perp\perp}$$

inherit « nice » properties.

Multiplicative completeness

Réservoir : $\S\mathbf{G} = \bigcup \P\mathbf{G}$.

Alienation : $\mathbf{G}, \mathbf{H}$ are alien when their reservoirs don't intersect :

$$\S\mathbf{G} \cap \S\mathbf{H} = \emptyset$$

Tensor $\otimes$: Notation in case of alienation ; the four tensors coincide.

Completeness :

$$\mathbf{G} \otimes \mathbf{H} = \{\mathfrak{A} \otimes \mathfrak{B}; \mathfrak{A} \in \mathbf{G}, \mathfrak{B} \in \mathbf{H}\}$$

Proof makes heavy use of weakening.

Quantifiers

Universal quantifier : Any intersection, think of intersection types, . . .

Prenex form : A quantifier commutes to every spiritual connective, but another quantifier.

$$\forall d(\mathbf{G}_d \oplus \mathbf{H}_d) = (\forall d\mathbf{G}_d) \oplus (\forall d\mathbf{H}_d)$$

Charles Péguy

Il faut toujours dire ce que l'on voit. Surtout il faut toujours, ce qui est plus difficile, voir ce que l'on voit.

Charles Péguy, Notre jeunesse, 1910.

Açores, 6 Septembre 2000

TO
LOSE
TO
WIN

Jean-Yves Girard

Example : the diagonal

First-order quantification : Not quite the conjunction $\varphi(\mathbf{G}) \,\&\, \psi(\mathbf{G})$. Indeed the diagonal $\{\varphi(\mathfrak{D}) \cup \psi(\mathfrak{D}); \mathfrak{D} \in \mathbf{G}\}$.

Exponential : Not quite the tensor product $\varphi(\mathbf{G}) \otimes \psi(\mathbf{G})$. Indeed the diagonal $\{\varphi(\mathfrak{D}) \otimes \psi(\mathfrak{D}); \mathfrak{D} \in \mathbf{G}\}$.

Incompleteness : Diagonals badly incomplete : biorthogonal equal to full with or full tensor !

Uniformity

Partial preorders : Transitive and weakly reflexive (PPR) :

$$x \preccurlyeq y \Rightarrow x \preccurlyeq x \text{ and } y \preccurlyeq y$$

Uniform designs : $\mathfrak{D} \preccurlyeq \mathfrak{D}$.

Diagonal : Equip With or Tensor with PPR
$\varphi(\mathfrak{D}') \cup \psi(\mathfrak{D}'') \preccurlyeq \varphi(\mathfrak{E}') \cup \psi(\mathfrak{E}'')$ (resp.
$\varphi(\mathfrak{D}') \otimes \psi(\mathfrak{D}'') \preccurlyeq \varphi(\mathfrak{E}') \otimes \psi(\mathfrak{E}'')$) iff

- $\mathfrak{D}' \preccurlyeq \mathfrak{D}'' \preccurlyeq \mathfrak{D}'$
- $\mathfrak{E}' \preccurlyeq \mathfrak{E}'' \preccurlyeq \mathfrak{E}'$
- $\mathfrak{D}' \preccurlyeq \mathfrak{E}'$

Weak behaviours

Weak designs : All designs (total or partial) included in some design of $\mathbf{G}$, notation $\mathfrak{D} \in \mathbf{G}^w$.

Incarnation :

$$|\mathfrak{D}|_{\mathbf{G}} = \bigcap \{|\mathfrak{E}| \,;\, \mathfrak{D} \subset \mathfrak{E} \in \mathbf{G}\}$$

Preorder Define $\mathfrak{D} \preccurlyeq_{\mathbf{G}} \mathfrak{D}' \Leftrightarrow |\mathfrak{D}|_{\mathbf{G}} \subset |\mathfrak{D}'|_{\mathbf{G}}$.

Duality In which sense is $(\mathbf{G}, \preccurlyeq_{\mathbf{G}})$ dual to $(\mathbf{G}^{\perp}, \preccurlyeq_{\mathbf{G}^{\perp}})$?

Bihaviours : The ground case is behaviour $\mathbf{G}$ equipped with the preorder $\preccurlyeq_{\mathbf{G}}$; everybody uniform in ground case.

Bihaviours

Bihaviour : Pair $(\mathbf{G}, \preccurlyeq)$ of a behaviour $\mathbf{G}$ and a PPR $\preccurlyeq$ on $\mathbf{G}^{w}$.

Constraints :

Positive base : If $\mathfrak{D} \preccurlyeq \mathfrak{E}$, then

- If $\mathfrak{E} = \mathfrak{Fid}$ then $\mathfrak{D} = \mathfrak{Fid}$.
- If $\mathfrak{D} \neq \mathfrak{Fid}$, then $\mathfrak{D} = \mathfrak{Dai}$ iff $\mathfrak{E} = \mathfrak{Dai}$.

Negative base : $\mathfrak{Sk} \preccurlyeq \mathfrak{Dai}^{-}$.

Orthogonality : $\mathfrak{D} \preccurlyeq \mathfrak{D}' \perp \mathfrak{E} \preccurlyeq \mathfrak{E}'$ iff

$$\ll \mathfrak{D} \mid \mathfrak{E} \gg \subset \begin{array}{c} \ll \mathfrak{D} \mid \mathfrak{E}' \gg \\ \ll \mathfrak{D}' \mid \mathfrak{E} \gg \end{array} \subset \ll \mathfrak{D}' \mid \mathfrak{E}' \gg$$

and

$$\ll \mathfrak{D} \mid \mathfrak{E} \gg = \ll \mathfrak{D} \mid \mathfrak{E}' \gg \cap \ll \mathfrak{D}' \mid \mathfrak{E} \gg$$

MALL$_2$

Cut

$$\frac{\Gamma \vdash \Delta ; P \quad \Gamma', P^l \vdash \Delta' ;}{\Gamma, \Gamma' \vdash \Delta, \Delta' ;}\ ;Cut \qquad \frac{\Gamma \vdash \Delta, P ; \quad \Gamma', P^l \vdash \Delta' ;}{\Gamma, \Gamma' \vdash \Delta, \Delta' ;}\ Cut;$$

Atom (Identity axiom)

$$\frac{}{\theta(X)^a \vdash ; \theta(X)}\ \vdash X$$

Focalisation

$$\frac{\Gamma^a \vdash \Delta ; P}{\Gamma^a \vdash \Delta, P ;}\ Foc$$

MALL$_2$

Shift

$$\frac{\Gamma^a, P^l \vdash \Delta;}{\Gamma^a \vdash \Delta; {\downarrow}P^\perp}\ \vdash{\downarrow} \qquad\qquad \frac{\Gamma \vdash \Delta, P;}{\Gamma, ({\downarrow}P^\perp)^l \vdash \Delta;}\ {\downarrow}\vdash$$

Zero

$$\frac{}{\Gamma, 0_{\langle 1 \rangle} \vdash \Delta;}\ 0\vdash$$

Plus

$$\frac{\Gamma^a \vdash \Delta; P}{\Gamma^a \vdash \Delta; P \oplus Q}\ \vdash l\oplus$$

$$\frac{\Gamma^a \vdash \Delta; Q}{\Gamma^a \vdash \Delta; P \oplus Q}\ \vdash r\oplus$$

$$\frac{\Gamma, P \vdash \Delta; \qquad \Gamma, Q \vdash \Delta;}{\Gamma, P \oplus Q \vdash \Delta;}\ \oplus\vdash$$

MALL$_2$

Tensor

$$\frac{\Gamma^a \vdash \Delta; P \qquad \Gamma'^a \vdash \Delta'; Q}{\Gamma^a, \Gamma'^a \vdash \Delta, \Delta'; P \otimes Q}\ \vdash\otimes \qquad\qquad \frac{\Gamma, P, Q \vdash \Delta;}{\Gamma, P \otimes Q \vdash \Delta;}\ \otimes\vdash$$

Existence

$$\frac{\Gamma^a \vdash \Delta; P[Q/X]}{\Gamma^a \vdash \Delta; \exists X P}\ \vdash\exists \qquad\qquad \frac{\Gamma, P \vdash \Delta;}{\Gamma, \exists X P \vdash \Delta;}\ \exists\vdash^{a}$$

---

[a] X is not free in Γ, Δ.

Soundness

- To each formula A associate a bihaviour $\mathbf{A}$. To variables $X, Y, \ldots$ associate bihaviours $\mathbf{X}, \mathbf{Y}, \ldots$ of base $\vdash \langle\rangle$ such that $\emptyset \notin \P\mathbf{X}$.
- To each proof π of closed formula A associate design $\boldsymbol{\pi} \in \boldsymbol{A}$.
- Induction loading :

$$\boldsymbol{\pi} \in \bigcap_{\mathbf{X}} [\bigotimes \mathbf{\Gamma} \vdash \mathbf{\Delta}, \mathbf{\Sigma}]$$

.

- If stoup non-empty, then first positive rule on stoup.

Hypocrisy

Characterisation : Restricted to Π^1. Still, not every design comes from a proof ; losers, the laymen of logic.

Ideal : Some principle, usually out of reach.

Failure : Cannot work.

Irresponsibility : Not my fault, he (Opponent) is responsible !

Uniformity

Ideal : We play several designs, e.g. $\mathfrak{D}_1 \subset \mathfrak{D}'_1$, $\mathfrak{D}_2 \subset \mathfrak{D}'_2$. Interaction involves eight nets.

These eight nets should be equal.

Not guilty : They are distinct, but due to Opponent…

Summing up : Up to incarnation, the four designs are identical. Condition amounts at requiring that

$$\mathfrak{D} \preccurlyeq \mathfrak{D}$$

Obstination

Ideal : A dispute should never end.

Not guilty : They are finite, but it’s because of Opponent.

Summing up : Up to incarnation, no daimon ! You must be stubborn.

Parsimony

Ideal : No waste, every available focus should be used.

Not guilty : Impossible since you create more than you destroy. I didn't focus on ξ, but Opponent should have focused on λ before…

Summing up : Designs-dessin decorated with exact rules, i.e. without weakening.

Full completeness

Winner : Uniform, stubborn and parsimonious… the ideal citizen.

Result : If A is Π^1, if $\mathfrak{D} \in \mathbf{A}$ is winning and material, then

$$\mathfrak{D} = \boldsymbol{\pi}$$

for an appropriate proof π of A.

Comment : Possible to prefer losers, incompleteness… But result essential anyway to prove that the notion has been really caught. Ludics is quite the general logical space, out of syntax, beyond our prejudices.

Non uniformity

Multiplicative case :

1. Write the two uniform designs in (forall $\mathbf{X}$) $\rho(\mathbf{X}) \otimes \rho'(\mathbf{X}) \vdash; \theta(\mathbf{X}) \otimes \theta'(\mathbf{X})$, called identity and flip.
2. For any partition of $(\wp_f(\mathbb{N}) – \emptyset) \times (\wp_f(\mathbb{N}) – \emptyset)$ in two classes, define a (non-uniform) stubborn design that behaves as identity or flip depending on the pair (I, J). Are these designs parsimonious ?

Additive case :

1. Write the two uniform designs in (forall $\mathbf{X}$) $\rho(\mathbf{X}) \oplus \rho'(\mathbf{X}) \vdash; \theta(\mathbf{X}) \oplus \theta'(\mathbf{X})$, called identity and flip.
2. For any partition of $\wp_f(\mathbb{N}) – \emptyset$ in two classes, define a (non-uniform) stubborn design that behaves as identity or flip depending on I. Are these designs parsimonious ?

The proof

Affine logic : Better to prove completeness for uniform designs w.r.t. system admitting weakening and daimons.

Polymorphic lemma : If

$$\mathfrak{F} \in \bigcap_{\mathbf{X}} \rho(\mathbf{X}) \otimes \rho'(\mathbf{X}) \vdash ; \theta(\mathbf{X})$$

then (1) or (2)

$$\forall \mathbf{X}\ \forall \mathfrak{D}, \mathfrak{D}' \in \mathbf{X}\ \ [\![\mathfrak{F}, \rho(\mathfrak{D}) \otimes \rho'(\mathfrak{D}')]\!] = \theta(\mathfrak{D}) \tag{1}$$

or

$$\forall \mathbf{X}\ \forall \mathfrak{D}, \mathfrak{D}' \in \mathbf{X}\ \ [\![\mathfrak{F}, \rho(\mathfrak{D}) \otimes \rho'(\mathfrak{D}')]\!] = \theta(\mathfrak{D}') \tag{2}$$

Concretely $\mathfrak{F}$ is the weakened version of a fax.

La Barbichette

Açores, 6 Septembre 2000

Je te tiens
Tu me tiens
Par la barbichette
Le premier qui rira
Aura une tapette.