Between logic and quantic : a tract

Jean-Yves Girard
Institut de Mathématiques de Luminy, UPR 9016 – **CNRS**
163, Avenue de Luminy, Case 930, F-13288 Marseille Cedex 09

girard@iml.univ-mrs.fr

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Abstract

We present a quantum interpretation of the perfect part of linear logic, by means of quantum coherent spaces. In particular this yields a novel interpretation of the reduction of the wave packet as the expression of η-conversion, a.k.a, extensionality.

Acknowledgements : this work has been essentially carried out in October 2002, and issued as privately circulated notes in French. The sources were my recent *ludics*, [9], that I was trying to make "quantic" for a couple of years, in relation to my much older "geometry of interaction", [6], an explanation of logic in terms of Hilbert space operators. In Spring 2002, I got a definite jolt from the work of Selinger, [11], in particular his handling of "density matrices". This definite version benefited from discussion with colleagues interested in the interface with quantum physics, Ctirad Klimčík, Thierry Paul, and Richard Zekri. It also benefited from the intercession of St Augustine, an output of the discussions led inside the informal group LGC "la Logique comme Géométrie du Cognitif", whose aim is to reconsider various philosophical and methodological issues that were fumbled by the "linguistic turn" of last century, see the page `http://www.logique.jussieu/www.joinet`.

1 Introduction

1.1 What is the question ?

From the beginning, it has been clear that something should be clarified between logic and quantic, that there was a logico-physical puzzle. In such a delicate situation, the main question was to find the *right* question.

1.1.1 The punishment of nature

According to Herodotus (VII,35), a tempest destroyed the military bridges built by Xerxes over the Hellespont ; he decided to punish nature and to have the sea whipped. To some extent, this is what logicians wanted to do to quantum physics, to punish it for being "against common-sense". Among the untold things was surely the idea of a complete schizophrenia between nature and spirit : our beautiful minds were harboured by the wrong world and this was a mere accident. The logical accounts of quantum phenomenons were contrived *on purpose*, as in the notorious *quantum logic* ; the subliminal message being : "quantum or not, just a matter of encoding".

1.1.2 The failure of quantum logic

We remember what happened to quantum logic —or worse, we no longer remember. Technically speaking, the idea of replacing Boolean algebras with the lattice of closed subspaces of a Hilbert space is obviously wrong : there is a fine negation (the orthogonal complement), but nothing like a decent conjunction, in other terms there is no simple account of the intersection of two spaces in terms of operators : $\pi \cap \pi' = \pi \cdot \pi'$ only when π, π' commute. Worse, the expulsion of the Hilbert space in favor of abstract "orthomodular lattices" didn't bring much water in this desert.

Viewed from a distance, there was a methodological mistake. Boolean algebras are the truth values of classical logic, they are used as *semantics*, the external world, in opposition with *syntax*, which deals with us, as observers. Quantum logic wanted to keep the opposition semantics/syntax, and, inside the same mould, slightly alter the semantics, from something simple (Boolean algebras) to something artificial (the closed subspaces of a Hilbert). But if there is something that the quantum world refuses, this is this simple minded view of an external reality. The logician Frege thought that any expression was denoting something ; but the word "impulsion", denotes nothing in quantum physics, worse, if we want it to denote something, we are performing an irreversible damage. In other terms we cannot separate between the world and its observation.

This explains the failure of quantum logic. There is little to say about other attempted interpretations, for instance via Kripke models, which are sort of branching parallel universes. These structures are so floppy that they give us back what we want to see in them : they are indeed Loyola models, they obey *perinde ac cadaver.*

1.2 Augustine vs. Thomas

We reverse the paradigm. We don't consider quantum as "immoral", we no longer try to "tame" it through some do-it-yourself logic. On the contrary, we consider the quantum world as nice, natural, welcoming. So nice indeed that logic should be interpreted, given a new space of freedom, inside the quantum world. This program forces me to say a few worlds as to the opposition between *essentialism* and *existentialism*, between Thomas and Augustine, the respective fathers of these two opposite conceptions.

1.2.1 Logical essentialism

Logic is surely born essentialist. And the essentialist interpretation is still overwhelmingly dominant. Take for instance Tarski's definition of truth : "$A \wedge B$ is true iff A is true *and* B is true". The essence of conjunction is primitive, all you can do is to express conjunction in terms of a meta-conjunction... We can say the same about a subtler logician, Kreisel, who proposed to reinterpret Brouwer's existentialist paradigms inside a formal system *given in advance.* To sum up : there are preexisting logical *laws.* Logical artifacts, proofs, models,... are constructed accordingly to the law. The reward for obeying the law is that everything goes right.

1.2.2 Logical Augustinism

The weak point of essentialism is that, if everything goes right, it means that something could go wrong, but how is it possible when the artifacts always follow the law ? The Augustinian[1] approach would be to admit that artifacts like proofs are anterior to logical declarations. Such was the viewpoint of the intuitionistic school (Kolmogoroff, Heyting : proofs as functions), and it seems that Gödel shared this opinion. However, the technical contents remained low.

Quantum is rather on the Augustinian side. An electronic spin is neither up or down w.r.t. a given axis, say $\vec{Z}$. If we only admit spins in these specific states, then we follow the logical laws governing boolean operations. But nature may shuffle the cards, tilt the gyroscopes, so that our would-be boolean has no definite spin on axis $\vec{Z}$. In an essentialist approach, this is illegal, immoral, and the measure of the value on this axis is simply forbidden. But we know from quantum physics that this measurement can take place, and that it involves the process known as the *reduction of the wave packet*, see section 5.3 for a logical discussion. Anyway, it is plain that the quantum world follows no rule.

1.3 The input of intuitionistic and linear logics

1.3.1 Functional interpretations

Around 1930, an alternative explanation of logic was presented by Kolmogoroff and Brouwer's pen-holder, Heyting. Proofs were basically functions, e.g., "a proof of $A \Rightarrow B$ is a function from proofs of A to proofs of B". This (sloppy) definition supposed that, somewhere, lived functions which were anterior to logic. In the late sixties, the Curry-Howard isomorphism expounded the categorical aspects of logic (proofs as morphisms) "a proof of $A \Rightarrow B$ is a morphism from A to B". These interpretations gave more and more importance to the proof, seen as a program, independent of logic. The original essentialist pattern was eventually turned upside down : a proof of A becomes a *program* enjoying *specification* A.

1.3.2 Locativity

It is obvious that the same program can enjoy distinct specifications, this is known as *subtyping.* We shall encounter subtyping in this paper, namely the subtyping **Bool** $\subset$ **Spin** (a Boolean, i.e., an electronic spin up or down w.r.t. the axis $\vec{Z}$, is a spin, i.e., a general electronic spin). What is specific about **Spin** is that it contains as many *isomorphic* copies $\mathbf{Bool}_{\vec{A}}$ of **Bool** as we want (one for each point of the unit sphere S^2) ; the isomorphism is not difficult to explain as a *spiritual* property ; essentialism considers things *as they should be.* However, the fact that **Spin** is the reunion of all $\mathbf{Bool}_{\vec{A}}$ cannot be explained in this way. This has to do with things *as they are*, with their precise *location*, with their physical incarnation, so to speak. In Augustinian terms, the objects come with a precise location, and isomorphism is the result of an accident —or rather a voluntary *delocation.* Locativity can embody spiritualism, whereas the converse is wrong[2].

1.3.3 Linear logic and actions

The technical input of linear logic, see, e.g., [7], was to replace proofs as functions with *proofs as actions.* In the linear implication $A \multimap B$, the premise is destroyed. This *perfect* (or perfective) aspect of linear logic is an essential novelty, in harmony with quantum phenomenons, typically the fact that a measurement alters the current state. Linear logic contains also *imperfect* connectives, which are more "classical". They are not studied in this paper : they require infinite dimension but, since this work crucially depends on the convergence of the trace, their study has been postponed.

1.3.4 Polarity

Why is the implication $\forall\exists \Rightarrow \exists\forall$ is wrong ? The usual answer is that in $\forall x \exists y$ the y depends on x, whereas in $\exists y \forall x$, y is independent of x... This "answer" is as original as Tarski's definition of truth ; it would be more honest to say "it is like this, period". I propose an explanation, based on the concept of *polarity* (positive/negative). This major divide gradually emerged from computer science in the years 1990, especially in the work of Andreoli on proof-search, [1]. This notion roughly separates :

Positive	/	**Negative**
$\oplus, \otimes$	/	$\&, \parr$
active	/	passive
$\overset{\lim}{\rightarrow}$	/	$\overset{\lim}{\leftarrow}$
synchronous	/	invertible
ℓ^1	/	ℓ^∞
explicit	/	implicit
object	/	subject
wave	/	measurement

The basic discovery of Andreoli is that operations of the same polarity commute. When polarities differ, we only have post-commutation of positive : a group $+-$ can be replaced with a group $-+$ (like in usual life, it is easy to *postpone* a decision). This is why $\forall\exists \Rightarrow \exists\forall$ is wrong and $\exists\forall \Rightarrow \forall\exists$ is correct.

1.3.5 Program of work and achievements

Intuitionism brought "proofs as functions, linear logic proposed "proofs as actions". We propose to refine this paradigm into "proofs as quantum actions" : by this me means that a proof of an implication $A \multimap B$ is any sort of transformation mapping "waves of type A" into "waves of type B", among which we include pure unitary transformations as well as pure measurements. Following a successful logical pattern, such transformations should also be seen as "waves of type $A \multimap B$", not as sort of "super-operators", like in Selinger's paper [11].

Hence proofs will be interpreted by operators. These operators should contain as particular cases, the usual "density matrices" and also the usual wave transformations and wave reductions, also expressed by hermitian operators. The only essentialist (i.e., coming "from the hat") concession is the choice of various finite-dimensional Hilbert spaces, but this is only because our formulas diverge in infinite dimension, otherwise we would once for all fix a separable Hilbert space. The basic duality is expressed by the formula :

$$u \mathrel{\overset{|}{\sim}} v \quad \Leftrightarrow \quad 0 \leq \mathrm{tr}(uv) \leq 1 \tag{1}$$

It is to be observed that logic will define various orderings between hermitians, and that a proper symmetry, such as the flip $\sigma(x \otimes y) = y \otimes x$ might be declared positive. This is because our framework embodies not only waves, but also "negative" (in the sense of polarity) artifacts, i.e., wave transformations $h \rightsquigarrow uau^*$.
The extension to infinite dimension, in relation to the bosonic or fermionic behaviour of the imperfect (non-linear, traditional) part of logic, is very exciting. But it seems that it deserves another treatment.

2 Commutative coherent spaces

Coherent spaces are usually presented in terms of a *web*, i.e., a carrier $\mathbb{X}$ together with a reflexive and symmetric relation on $\mathbb{X}$, its *coherence*. We shall replace this *essentialist* approach, in which the coherence relation is primitive with an alternative existentialist, *Augustinian*, in which coherence is the result of interaction. The starting remark is that we are basically interested in *cliques*, i.e., coherent subsets of the carrier, and that the negation deals with anti-cliques, i.e., incoherent subsets, so that a clique and an anti-clique intersect on at most one point[3].

2.1 Revisiting coherent spaces

Definition 1
Let $\mathbb{X}$ be a set ; two subsets $a, b \subset \mathbb{X}$ are polar *when their intersection is at most a singleton. In notations*

$$a \mathrel{\overset{|}{\sim}} b \quad \Leftrightarrow \quad \sharp(a \cap b) \leq 1 \tag{2}$$

We define the polar $\sim A$ *of a set $A \subset \wp(\mathbb{X})$ of subsets of $\mathbb{X}$ by :*

$$b \in \sim A \quad \Leftrightarrow \quad \forall a \in A \quad a \mathrel{\overset{|}{\sim}} b \tag{3}$$

A coherent space *with* carrier $\mathbb{X}$ *is a subset $X \subset \wp(\mathbb{X})$ equal to its bipolar. Equivalently, a coherent space is the polar of some subset ; moreover the map $X \rightsquigarrow \sim X$ is an involution of coherent spaces with carrier $\mathbb{X}$, the (linear)* negation.

The fact that we make a heavy use of Hilbert spaces prompts us to adapt the terminology and notations of linear logic : orthogonality, $\perp$ and $A^{\perp}$ are replaced with polarity, $\underset{\sim}{\mid}$ and $\sim A$.
Let X be a coherent space with carrier $\mathbb{X}$:

- *(i)* X contains the empty set and all singletons $\{x\}$ $(x \in \mathbb{X})$; in particular, X is not empty.
- *(ii)* If $a' \subset a \in X$, then $a' \in X$: this is because $\sharp(a' \cap b) \leq \sharp(a \cap b)$.
- *(iii)* If $a \subset \mathbb{X}$ and $a \notin X$, there are $x, y \in X, x \neq y$ such that $\{x, y\} \notin X$: if $b \in \sim X$ is such that $\sharp(a \cap b) \geq 2$, let x, y be two distinct elements of $a \cap b$.
- *(iv)* If $x, y \in \mathbb{X}$ are distinct, then $\{x, y\} \notin X$ iff $\{x, y\} \in \sim X$: obviously $\{x, y\}$ cannot belong to both, moreover, if $\{x, y\} \notin \sim X$, this means that some $a \in X$ contains two distinct points of $\{x, y\}$.

This suggests the following definition :

Definition 2
If X is a coherent space with carrier $\mathbb{X}$, we define a binary relation on $\mathbb{X}$, coherence *:*

$$x \frown\!\!\!\!\smile_X y \quad \Leftrightarrow \quad \{x, y\} \in X \tag{4}$$

By what precedes, coherence w.r.t. $\sim X$, $\frown\!\!\!\!\smile_{\sim X}$, is identical to incoherence *w.r.t. X :*

$$x \smile\!\!\!\!\frown_X y \quad \Leftrightarrow \quad x = y \vee x \not\frown\!\!\!\!\smile_X y \tag{5}$$

The following proposition establishes the equivalence between definition 1 and definition 2, the original definition of coherent spaces.

Proposition 1
Let X be a coherent space with carrier $\mathbb{X}$, and let $a \subset \mathbb{X}$. Then $a \in X$ iff a is a clique *w.r.t. the coherence of X, namely, if $\forall x, y \in a \quad x \frown\!\!\!\!\smile_X y$.*

Proof : Immediate. $\square$

2.2 Perfection vs. imperfection

Logic can be interpreted in coherent spaces : a formula become a coherent spaces and its proofs become elements (cliques) in it, see, e.g., [7]. Originally, coherent spaces were intended as an explanation of *intuitionistic* logic. The main achievement was to interpret intuitionistic (*imperfect*, see below) implication $X \Rightarrow Y$ in two equivalent ways : either by means of functions from X to Y or by means of a coherent space $X \Rightarrow Y$. $X \Rightarrow Y$ turned out to be a compound operation, made out of two primitives, $\multimap$ and ! :

$$X \Rightarrow Y \quad = \quad !X \multimap Y \tag{6}$$

The linear implication $\multimap$ is *causal*, in the sense that, in a linear implication, the premise cannot be reused : $X \multimap Y$, enables one, given (a clique in) X, to get (a clique in) Y, but the premise is destroyed. If one wants to reuse the premise, one has to say something like "forever X", which involves the construction of the *exponential* $!X$.
The main achievement of linear logic was not quite to change logical connectives and rules, but to distinguish a primal *linear* layer, in which things are *performed* once for all, that one should therefore called *perfect*, in analogy with linguistics : perfect tenses are used to denote a punctual, well-defined action ; in French, English, this is limited to the past, in Russian, this is more systematic. Perfect connectives come as *dual* pairs, $\oplus/\&$, $\otimes/\invamp$; duality means that each pair is swapped by linear (perfect) negation, e.g., $\sim(X \otimes Y) = \sim X \invamp \sim Y$. The most important connective is not part of this official list : *linear* (perfect) implication $X \multimap Y$ is indeed $\sim X \invamp Y$.
Imperfection corresponds to general statements, e.g., mathematical theorems, or to repetitive actions. James Bond movies often have imperfect titles "Diamonds are forever", "You only live twice" (compare to perfect titles like "Gunfight at the OK Corral" !). Imperfect implication $\Rightarrow$ does not correspond to linear maps, but rather to analytical maps, see [8][4]. Mathematically speaking, imperfection deals with infinity, whereas perfection can reasonably live in a small (finite) world. This has a consequence for this paper : *quantum coherent spaces* make a heavy use of the *trace* which (basically) lives in finite-dimensional spaces. This means that we shall forget the imperfect connectives !/? which would involve infinite dimension and concentrate on the perfect $\oplus/\&$, $\otimes/\invamp/\multimap$. Since this paper is basically concerned with the relation logic/quantum, this is not a major restriction : perfection is rather "quantum" whereas imperfection is more "classical".

2.3 Perfect connectives

The basic perfect connectives are divided into *additives* and *multiplicatives* ; additives make use of disjoint *unions* (later : direct sums), multiplicatives make use of cartesian *products* (later : tensor products).

2.3.1 Additives

Assume that the respective carriers $\mathbb{X}$, $\mathbb{Y}$ of X, Y are disjoint (if not, do the obvious thing !). Then we define $X \oplus Y$, *"Plus"*, and $X \,\&\, Y$, *"With"*, both with carrier $\mathbb{X} \cup \mathbb{Y}$:

Definition 3
$\oplus$ and $\&$ are defined by the dual definitions :

$$X \oplus Y = X \cup Y \tag{7}$$

$$X \,\&\, Y = \{a \cup b \,; a \in X, b \in Y\} \tag{8}$$

Proposition 2
$X \oplus Y$ and $X \,\&\, Y$ are coherent spaces ; their respective negations are $\sim X\& \ \sim Y$ and $\sim X \oplus \ \sim Y$.

Proof : Everything eventually amounts at showing that the spaces $X \oplus Y$ and $\sim X\& \ \sim Y$ are swapped by negation. Any $c \subset \mathbb{X} \cup \mathbb{Y}$ can uniquely be written $c = a \cup b$, with $a \subset \mathbb{X}$, $b \subset \mathbb{Y}$. $c = a \cup b \in \sim(X \oplus Y)$ iff $c \underset{\sim}{\downarrow} a'$ and $c \underset{\sim}{\downarrow} b'$ for all $a' \in X$, all $b' \in Y$, i.e., iff $c \in \sim X\& \ \sim Y$, which shows that $\sim(X \oplus Y) = \sim X\& \ \sim Y$.
From this we deduce that $X \oplus Y \subset \sim\sim(X \oplus Y) = \sim(\sim X\& \ \sim Y)$. But if $c = a \cup b \in \sim(\sim X\& \ \sim Y)$, one of a, b must be empty : if $x \in a \subset \mathbb{X}, y \in b \subset \mathbb{Y}$, then $\{x, y\} \in \sim(\sim X\& \ \sim Y)$, and $\neg(c \underset{\sim}{\downarrow} \{x, y\})$. Let us say that $c = b$; then c meets any $a' \cup b'$ $(a' \in \sim X,\ b' \in \sim Y)$ on at most one point, which means that $b \underset{\sim}{\downarrow} b'$, and that $c \in Y$. From this, $X \oplus Y = \sim(\sim X\& \ \sim Y)$. $\square$

The coherence relations related to additives work as follows : if $x, x' \in \mathbb{X}$, then $x \stackrel{\frown}{\smile} x'$ w.r.t. $X \oplus Y$ or $X \,\&\, Y$ iff they were coherent w.r.t. X, similarly for $y, y' \in \mathbb{Y}$. The connectives differ as to the coherence between $x \in \mathbb{X}$ and $y \in \mathbb{Y}$:

$X \oplus Y$ **:** incoherent, $x \stackrel{\smile}{\frown} y$.

$X \,\&\, Y$ **:** coherent, $x \stackrel{\frown}{\smile} y$.

2.3.2 Multiplicatives

Assume that the respective carriers of X and Y are $\mathbb{X}$ and $\mathbb{Y}$. Then we define $X \otimes Y$, *"Times"*, and $X \parr Y$, *"Par"*, both with carrier $\mathbb{X} \times \mathbb{Y}$; we start with the essentialist version (via coherence), and later discuss the possibility of an existentialist version. The following abbreviations are useful : $x \frown y$ for

$x \overset{\frown}{\smile} y \wedge x \neq y$, $x \smile y$ for $x \overset{\smile}{\frown} y \wedge x \neq y$ (equivalently, $x \frown y \Leftrightarrow x \not\overset{\smile}{\frown} y$, $x \smile y \Leftrightarrow x \not\overset{\frown}{\smile} y$).

Definition 4
The respective coherences of "Times" and "Par" are as follows :
in $X \otimes Y$, $(x, y) \overset{\frown}{\smile} (x', y')$ *iff* $x \overset{\frown}{\smile} x'$ and $y \overset{\frown}{\smile} y'$.
in $X \parr Y$, $(x, y) \frown (x', y')$ *iff* $x \frown x'$ or $y \frown y'$.

The two definitions are clearly dual ; "Par" is an artificial creation, the dual of "Times". Indeed "Par" is a contrived way to speak of *linear* implication, $X \multimap Y = \sim X \parr Y$, and $X \parr Y$ is better understood as $\sim X \multimap Y$ or $\sim Y \multimap X$. The coherence on $X \multimap Y$ is obviously given by :
$(x, y) \frown (x', y')$ iff $x \overset{\frown}{\smile} x' \Rightarrow y \frown y'$.

Definition 5
A function φ from (cliques of) X to (cliques of) Y is linear *when it preserves all disjoint unions : if a_i are disjoint cliques in X whose union is still a clique, then*

$$\varphi(\bigcup_i a_i) = \bigcup_i \varphi(a_i)$$

The following result is elementary, but essential :

Theorem 1
If A is a clique in $X \multimap Y$ and a is a clique in X, define

$$[A]a := \{y \in \mathbb{Y}; \exists x \in a \quad (x, y) \in A\} \tag{9}$$

Then $[A]a$ is a clique in Y and the map $a \rightsquigarrow [A]a$ is linear.
Moreover, any linear function φ from X to Y is of the form $[A]\cdot$, with a unique A given by :

$$A = \{(x, y); y \in \varphi(\{x\})\} \tag{10}$$

Proof : See the literature, e.g., [7]. The crucial point in the proof is the fact that in (9), the x such that $(x, y) \in A$ is indeed unique. □

Example 1
Since linearity is a preservation property, the identity map is surely linear. The clique in $X \multimap X$ associated to it is the set $\Delta_{\mathbb{X}} = \{(x, x) \; ; x \in \mathbb{X}\}$. This set is not the graph $\{(a, a) \; ; a \in X\}$ of the function, it is much smaller, and depends only on the carrier $\mathbb{X}$.

The theorem establishes a link between the cliques of the coherent space $X \multimap Y$ and the linear functions from X to Y ; we could as well take linear functions ψ from $\sim Y$ to $\sim X$, using $b \rightsquigarrow b[A]$:

$$b[A] := \{x \in \mathbb{X} \; ; \; \exists y \in b \;\; (x,y) \in A\} \tag{11}$$

and

$$A = \{(x,y) \; ; \; x \in \psi(\{y\})\} \tag{12}$$

Let us now try an Augustinian definition of multiplicatives. There is no problem as long as $\otimes$ is concerned :

Proposition 3
$X \otimes Y = \{c \; ; \; \exists a \in X \; \exists b \in y \quad c \subset a \times b\}$.

Proof : If $a \in X$, $b \in Y$, then $a \times b \in X \otimes Y$, and if $c \subset a \times b$, we still have $c \in X \otimes Y$. Conversely, if $c \in X \otimes Y$, let a, b be the respective projections of c on $\mathbb{X}$ and $\mathbb{Y}$; then $c \subset a \times b$ and $a \in X$, $b \in Y$. $\square$

But there is nothing of the like for the connective "Par", or equivalently, linear implication. However the following is true :

Proposition 4
$X \multimap Y = \{A \; ; \; \forall a \in X \;\; [A]a \in Y\}$.

Proof : Trivial reformulation of theorem 1. $\square$

Moreover, $[A]a$ is characterised as the unique subset of $\mathbb{Y}$ such that :

$$\sharp([A]a \;\; \cap \;\; b) = \sharp(A \;\; \cap \;\; a \times b) \tag{13}$$

for any $b \in \sim Y$, so what is the problem ? Following the existentialist pattern, existence (here : objects, functions) must be anterior to essence (here : logical declarations). This means that we should be able to define $[A]a$ for any subsets $A \subset \mathbb{X} \times \mathbb{Y}$, $a \subset \mathbb{X}$, in such a way that (13) holds for all $b \subset \mathbb{Y}$. But this is clearly impossible : we have an explicit definition of $[A]a$ in (9), and it is plain that (13) is satisfied iff the x such that $(x,y) \in A$ is unique. Our construction is essentialist in the sense that $[A]a$ is defined only when A, a "obey the law".

You may think that I am gilding the lily, asking for some fancy purity criteria... And this is correct as long as we stay with usual (commutative) spaces : everything can be handled in terms of a well-defined set of atoms (the singletons $\{x\}, x \in \mathbb{X}$). But imagine that the atoms are no longer well-defined (no canonical base in a vector space), or, worse, that there are no atoms at all (e.g. in a von Neumann algebra of type distinct from I). By the way, in what follows (PCS, QCS), there will be no direct, manageable, account of the tensor product in the style of proposition 3, and our only hope will be the linear implication.

2.4 Probabilistic coherent spaces

In proceeding towards quantum, we must replace qualitative features with quantitative ones. Here it is the place to remark that my first glimpse of linear logic came from *quantitative* domains, [5], see also [2], soon replaced with qualitative domains and coherent spaces. Indeed there is something quantitative in coherent spaces, namely the unicity of the x in (9), which is behind (13).

The idea will therefore to replace $\wp(\mathbb{X})$ —the subsets of the carrier $\mathbb{X}$— with the space $\Re(\mathbb{X})$ of all functions $\mathbb{X} \xrightarrow{f} \mathbb{R}^+$. We have in mind that, instead of saying whether or not $x \in \mathbb{X}$ belongs to a set, we rather give a probability, which would mean $0 \leq f \leq 1$; incoherence between two atoms x, y now means that their mutual weights $f(x), f(y)$ are such that $f(x) + f(y) \leq 1$, which amounts to a mutual exclusion, in case f is a characteristic function. But this is only a basic intuition : once for all, forget about coherence, or any limitation of the values to the interval $[0, 1]$.

2.4.1 The bipolar theorem

Definition 6
Let $\mathbb{X}$ be a finite set ; two functions $f, g : \mathbb{X} \rightarrow \mathbb{R}^+$ are polar *when :*

$$\sum_{x \in \mathbb{X}} f(x) \cdot g(x) \leq 1 \tag{14}$$

We define the polar of a set of positive functions as in definition 1, and a probabilistic *coherent space (PCS) as a set of positive functions equal to its bipolar.*

(14) is obviously inspired from (2), since, when f, g are the characteristic functions of the subsets $a, b \subset \mathbb{X}$, then $\sum_{x \in \mathbb{X}} f(x) \cdot g(x) = \sharp(a \cap b)$.

Theorem 2 (Bipolar theorem)
Let X be a PCS with carrier $\mathbb{X}$; then

(i) *X is non-empty (in fact, $0_{\mathbb{X}}$ belongs to X).*

(ii) *X is closed and convex.*

(iii) *X is downward closed.*

Conversely every subset of $\Re(\mathbb{X})$ enjoying (i)-(iii) is a PCS.

Proof : That every PCS enjoys (i)-(iii) is a trifle. Conversely, assume that X enjoys (i)-(iii) and that $f \notin X$. $\Re(\mathbb{X})$ is a closed convex subset of the real Banach space $\mathbb{R}^{\mathbb{X}}$. By Hahn-Banach, there is a linear form φ such that $\varphi(X) \leq 1$, $\varphi(f) > 1$. This linear form can be identified with an element of $\mathbb{R}^{\mathbb{X}}$, i.e., a real-valued function h : $\varphi(g) = \sum_{x \in \mathbb{X}} g(x) \cdot h(x)$. Define the positive h' by $h'(x) = \sup(h(x), 0)$. Obviously $\sum_{x \in \mathbb{X}} h'(x) \cdot f(x) \geq \sum_{x \in \mathbb{X}} h(x) \cdot f(x) > 1$. If $g \in X$, then $\sum_{x \in \mathbb{X}} h'(x) \cdot g(x) = \sum_{x \in \mathbb{X}} h(x) \cdot g'(x)$ with $g'(x) = 0$ if $h'(x) = 0$, $g'(x) = g(x)$ otherwise ; $g' \leq g \in X$, hence $g' \in X$ by (iii), and $\sum_{x \in \mathbb{X}} h'(x) \cdot g(x) \leq 1$. This shows that $h' \in {\sim} X$, but $\neg(h' \underset{\sim}{\downarrow} f)$, hence $f \notin X$. □

2.4.2 Additives

As before, additives are defined in case the carriers $\mathbb{X}$ and $\mathbb{Y}$ of X, Y are disjoint, as a PCS with carrier $\mathbb{X} \cup \mathbb{Y}$. If $f \in \Re(\mathbb{X})$, $g \in \Re(\mathbb{Y})$, I can define $f \cup g \in \Re(\mathbb{X} \cup \mathbb{Y})$ in the obvious way. The set

$$X \,\&\, Y := \{f \cup g \ ; f \in X, g \in Y\} \tag{15}$$

is the polar of ${\sim}X \cup {\sim}Y$ (modulo the obvious abuse which identifies $f \in \Re(\mathbb{X})$ with $f \cup 0_{\mathbb{Y}} \in \Re(\mathbb{X} \cup \mathbb{Y})$, so that ${\sim}X \cup {\sim}Y$ is indeed short for $\{f \cup g \ ; f \in {\sim}X, g \in {\sim}Y, f = 0 \vee g = 0\}$). On the other hand $X \cup Y$ is not a PCS ; $X \oplus Y$ must be defined as ${\sim}{\sim}(X \cup Y)$, with no hope of removing the double negation. The bipolar theorem 2 yields :

Proposition 5

$$X \oplus Y = \{\lambda f \cup (1 - \lambda) g \ ; f \in X, g \in Y, 0 \leq \lambda \leq 1\} \tag{16}$$

Proof : The right-hand side is the convex hull of $X \cup Y$. It obviously enjoys conditions (i)-(iii). □

2.4.3 Multiplicatives

As before, multiplicatives are defined as PCS with carrier $\mathbb{X} \times \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are the respective carriers of X, Y. But, in contrast with section 2.3.2, the definition is really Augustinian.

Definition 7
If $\Phi \in \Re(\mathbb{X} \times \mathbb{Y})$*, if* $f \in \Re(\mathbb{X})$*, then one defines* $[\Phi]f \in \Re(\mathbb{Y})$ *by :*

$$([\Phi]f)(y) = \sum_{x \in \mathbb{X}} \Phi(x, y) \cdot f(x) \tag{17}$$

This makes sense because $\mathbb{X}$ is finite.

Theorem 3
The map $\Phi \rightsquigarrow [\Phi]\cdot$ *is a bijection from* $\Re(\mathbb{X} \times \mathbb{Y})$ *onto the set of linear maps from* $\Re(\mathbb{X})$ *to* $\Re(\mathbb{Y})$*.* Φ *can be retrieved from its associated linear map* $\varphi = [\Phi]\cdot$ *by means of :*

$$\Phi(x, y) = \varphi(\delta_x)(y) \tag{18}$$

Proof : A linear map satisfies $\varphi(\lambda f + \mu g) = \lambda\varphi(f) + \mu\varphi(g)$, hence it is determined by its values on the δ_x, this is the explanation of equation (18). Everything is straightforward. ☐

In the basic case (subsets) this didn't work : if Φ and f are characteristic functions, $[\Phi]f$ need not be one (again the unicity of the x in (9)).

Definition 8
If X, Y *are PCS with respective carriers* $\mathbb{X}, \mathbb{Y}$*, one defines the PCS* $X \multimap Y$*, with carrier* $\mathbb{X} \times \mathbb{Y}$*, as the set of all* Φ *such that* $[\Phi]\cdot$ *maps* X *to* Y*.*

Example 2
The characteristic function $\Delta_{\mathbb{X}}$ *of the diagonal belongs to* $X \multimap X$ *; in fact* $[\Delta_{\mathbb{X}}]f = f$*.*

$X \multimap Y$ is obviously the polar of $\{f \times g \,; f \in X, g \in {\sim} Y\}$, this why it is a PCS. It could as well be defined as the set of all Φ such that $\cdot[\Phi]$ (whose definition is easy to figure out) sends ${\sim}Y$ to ${\sim}X$.
From $\multimap$, "Par" and "Times" follow, e.g., $X \otimes Y := {\sim}(X \multimap {\sim}Y)$, equivalently, $X \otimes Y = {\sim}{\sim}\{f \times g; f \in X, g \in Y\}$. The bipolar theorem characterises this set as a closure under certain operations, but this is not very manageable. Should we try to prove associativity of "Times", it is much simpler to first establish it for the dual connective $\parr$.

Proposition 6
"Par" is commutative, associative, and distributes over "With".

Proof : Let us prove, for instance, that "Par" is associative. For this, we pretend that cartesian product is really[5] associative, so that we can write $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ as the common carrier of $X \parr (Y \parr Z)$ and $(X \parr Y) \parr Z$. We use the possibility of expressing "Par" in two ways, by means of $[\cdot]\cdot$ or $\cdot[\cdot]$. We get $X \parr (Y \parr Z) = \{A \ ; \ \forall f \in {\sim}X \ \forall h \in {\sim}Z \quad h[[A]f] \in Y\}$, whereas $(X \parr Y) \parr Z = \{A \ ; \ \forall h \in {\sim}Z \ \forall f \in {\sim}X \quad [h[A]]f \in Y\}$. Everything amounts at checking that $h[[A]f] = [h[A]]f$, which is obvious.
Similarly, if we want to prove that "Par" distributes over "With", by establishing an isomorphism between $X \parr (Y \,\&\, Z)$ and $(X \parr Y) \,\&\, (X \parr Z)$, we express "Par" in terms of $[\cdot]\cdot$ (and not in terms of $\cdot[\cdot]$, which is suitable for distribution on the left). ☐

3 Generalisations

3.1 Köthe spaces

The restriction to finite carriers ensures the convergence of (18). In the case of infinite carriers, one can liberalise the definition so as to accept the value $+\infty$. One can also use *Köthe spaces* : the objects are functions from a carrier $\mathbb{I}$ to $\mathbb{R}$, and polarity is defined by :[6]

$$f \mathrel{\overset{|}{\sim}} g \quad \Leftrightarrow \quad \sum_{i \in \mathbb{I}} |f(i) \cdot g(i)| \leq +\infty \tag{19}$$

This is what Ehrhard did in [4] ; in that case, (18) does not always make sense. However, everything works fine, as long as one does not try to "change the basis", i.e., as long as one "stays commutative".

3.2 Continuous carriers : an interesting failure

There seems to be an alternative way to accommodate infinite carriers, namely, to consider $\mathbb{X}$ as a *measure space*, typically the segment $[0, 1]$ with Lebesgue measure. It will turn out that this attempt fails, but sometimes a wrong idea is far more interesting than a "correct" one. We only sketch the definitions :

Carriers : measure spaces $(\mathbb{X}, \mu)$, $\mathbb{X}$ for short.

Objects : functions $\mathbb{X} \xrightarrow{f} \mathbb{R}^+$ which are essentially bounded, i.e., $f \in \mathcal{L}^\infty(\mathbb{X}, \mathbb{R}^+)$.

Polarity : $f \mathrel{\overset{|}{\sim}} g \quad \Leftrightarrow \quad \int_{\mathbb{X}} f \cdot g \; d\mu \leq 1$.

Application : given Φ, f with respective carriers $\mathbb{X}\times\mathbb{Y}$ and $\mathbb{X}$, define $[\Phi]f$, with carrier $\mathbb{Y}$, by $([\Phi]f)(y) = \int_{\mathbb{X}} \Phi(x,y)\cdot f(x)\ d\mu(x)$.

The map $\Phi \rightsquigarrow [\Phi]\cdot$ associates to each Φ with carrier $\mathbb{X}\times\mathbb{Y}$ a linear map sending objects with carrier $\mathbb{X}$ to objects with carrier $\mathbb{Y}$. Unfortunately, this map is far from being surjective, the typical example being given by the identity map (in case $\mathbb{X} = \mathbb{Y}$). The obvious candidate for this is still Δ, the characteristic function of the diagonal, see example 2. But this function is likely to be almost everywhere null. This is where we fail, and we shall meet the same obstacle when dealing with QCS.
This failed attempt introduced an important novelty, namely that the basic duality should be seen as an *integral* (remember that we started with an intersection). Since, following Connes, the non-commutative integral is a trace, this explains the role played by the trace in a QCS.

3.3 Banach spaces

In [8], I introduced *coherent Banach spaces* as an explanation for logic. These spaces were complex because of the use of analytic functions in the imperfect case ; they were also infinite dimensional, which forced one to be careful with problems of *reflexivity*. Here we restrict our discussion to real, finite dimensional, Banach spaces.

Norms Banach spaces are normed : X is a finite dimensional real Banach space, and $\sim X$ is its dual, with dual norm, so that the identification $X = \sim\sim X$ makes sense. But what is this norm for ? The answer is that the norm measures *incoherence*, what corresponds to cliques of a coherent space, to objects of a PCS, is now translated as a vector of norm ≤ 1.

Additives The underlying space is a direct sum $X \oplus Y$, only the norms differ :

$$\|f \oplus g\|_{X\oplus Y} = \|f\|_X + \|g\|_Y \tag{20}$$
$$\|f \oplus g\|_{X\& Y} = \sup(\|f\|_X, \|g\|_Y) \tag{21}$$

The two choices are dual.

Multiplicatives $X \multimap Y$ is the space of linear maps from X to Y, endowed with the usual supremum norm. $X \otimes Y$ is the tensor product, endowed with the usual tensor norm, defined as $\|a\|_{X\otimes Y} = \inf\{\sum_i \|x_i\|_X \cdot \|y_i\|_Y\}$, the infimum being taken over all decompositions $a = \sum_i x_i \cdot y_i$. Again the two choices are dual.

Polarity Certain norms are defined via supremum, this is the case for & and $\multimap$ (i.e., $\parr$), others in terms of sums $(\oplus, \otimes)$. The choice of supremum corresponds to coherence, the choice of sum to incoherence. This distinction is a major divide of logic, known as *polarity*, see the introduction : supremum is negative (observation-like), sum is positive (object-like).

Semi-norms There is *a priori* no room for semi-norms in this picture. In usual mathematics, a semi-norm behaves like a norm on a quotient space. However this is wrong in the case of logic, especially if we want to accommodate Augustinian features such as subtyping. The subtyping $X \subset Y$ means that, on the same underlying vector space $\mathbb{E}$, we can have more "coherent" objects, i.e., that the unit ball increases. In other terms, $\|\cdot\|_Y \leq \|\cdot\|_X$: the norm decreases. It can decrease up to 0 on certain vectors, and this explains why semi-norms naturally occur.

Positivity PCS were made of positive functions, hence they were an ordered structure. The same is true of real Köthe spaces, which are spaces of sequences. With Banach spaces, things are different, since there is *a priori* no distinguished basis. However, observe the following property :

Proposition 7
$f \in \mathbb{R}^{\mathbb{X}}$ belongs to $\Re(\mathbb{X})$ iff for all $g \in \Re(\mathbb{X})$ the "scalar product" $\sum_{x \in \mathbb{X}} f(x) \cdot g(x)$ is positive.

Proof : Immediate. $\square$

This means that positivity itself can be defined in Augustinian style. We shall make a heavy use of this when dealing with QCS... although QCS are spaces of hermitian operators, which come with a natural ordering (positive hermitians), we shall not content ourselves with the "standard" notion of positivity. This can be very easily understood : if $\|a\| = 0$ and $a \neq 0$, then it is reasonable to assume that a can be identified with 0, which means that $0 \leq a$ and $0 \leq -a$. The a and $-a$ cannot both be positive hermitians.

3.4 The bipolar theorem, revisited

We shall complete our preliminary works with an alternative version of the bipolar theorem 2 which requires some care. The setting is as follows : $\mathbb{E}$ is a finite-dimensional Euclidian space, equipped with the bilinear form $\langle \cdot \mid \cdot \rangle$. Polarity is defined by means of :

$$x \overset{|}{\sim} y \quad \Leftrightarrow \quad 0 \leq \langle x \mid y \rangle \leq 1 \tag{22}$$

The question is to determine bipolars.

Theorem 4 (Bipolar theorem)
A subset $C \subset \mathbb{E}$ is its own bipolar iff the following hold :

(i) $0 \in C$.

(ii) *C is closed and convex.*

(iii) *If $nx \in C$ for all $n \in \mathbb{N}$, then $-x \in C$.*

(iv) *If $x, y \in C$ if $\lambda, \mu \geq 0$ and $\lambda x + \mu y \in C$, then $\lambda x \in C$.*

Proof : (i) and (ii) are immediate. (iii) : if $nx \in C$ for $n \in \mathbb{N}$, and $z \in {\sim}C$, then $\langle x \mid z \rangle \in [0, 1/n]$ for $n \in \mathbb{N}$, hence $\langle -x \mid z \rangle = \langle x \mid z \rangle = 0 \in [0, 1]$.
(iv) : if $z \in {\sim}C$, then $0 \leq \langle \lambda x + \mu y \mid z \rangle \leq 1$, $0 \leq \langle \lambda x \mid z \rangle$, $0 \leq \langle \mu y \mid z \rangle$, hence $0 \leq \langle \lambda x \mid z \rangle \leq 1$. By the way observe that (iv) yields a sort of converse to (iii) : if $x, -x \in C$, then $nx + n(-x) = 0 \in C$, hence $nx \in C$.
We now prove the converse, and assume that C enjoys (i)-(iv) ; let C^+ be the cone $\bigcup_{n \in \mathbb{N}} n \cdot C$ $(= \bigcup_{\lambda \in \mathbb{R}^+} \lambda \cdot C)$. Then we can reformulate (iv) as :

$$C = C^+ \cap (C - C^+) \tag{23}$$

Assume that $b \notin C$, then, by (23), we have to consider two cases :

$b \notin C^+$: using Hahn-Banach, one can find a vector $d \in \mathbb{E}$ such that $\langle b \mid d \rangle < 0 \leq \langle c \mid d \rangle$ for all $c \in C$. By condition (ii) the subset $I = \{c \,;\, \forall n \in \mathbb{N} \;\; nc \in C\}$ is a vector space, moreover, $\langle \cdot \mid d \rangle$ vanishes on I, so that we can write $C = I \oplus C'$, with $C' = I^\perp \cap C$. C' is compact : if we embed $\mathbb{E}$ in the projective space, C' has a compact closure, and its boundary corresponds to the lines $\mathbb{R} \cdot a$ which are included in C'. But there is no such line (all of them have been gathered in I) : the boundary is empty, and C' is compact. From this, $\langle \cdot \mid d \rangle$ is bounded on C', hence on C, so $\langle b \mid d \rangle < 0 \leq \langle c \mid d \rangle \leq \lambda$. By rescaling d we can assume that $\lambda = 1$, in which case $d \in {\sim}C$, and $b \notin {\sim}{\sim}C$.

$b \notin C - C^+$: the same Hahn-Banach yields a vector $d \in \mathbb{E}$ such that $\langle p \mid d \rangle \leq 1 < \langle b \mid d \rangle$, for all $p \in C - C^+$. Assume that $\langle c \mid d \rangle < 0$ for some $c \in C$; then $-nc \in C - C^+$ for $n \in \mathbb{N}$ and the values $\langle -nc \mid d \rangle$ cannot be bounded by 1. From this we deduce that $0 \leq \langle c \mid d \rangle \leq 1 < \langle b \mid d \rangle$ for all $c \in C$. As above, $d \in {\sim}C$, and $b \notin {\sim}{\sim}C$.

□

3.5 Norm and order

With the notations of theorem 4, in particular, $D = \sim C, C^+ = \bigcup_{n\in\mathbb{N}} n \cdot C$:

Definition 9
The domain Fin_C *of* C *is the vector space* $C^+ - C^+$ *generated by* C.

Proposition 8
$\mathrm{Fin}_C = (D \cap (-D))^\perp$.

Proof : If $c \in C, d \in D \cap (-D)$, then $\langle c \mid d\rangle = 0$, and the same remains true for $c \in \mathrm{Fin}_C$, the linear span of C, so that $\mathrm{Fin}_C \subset (D \cap (-D))^\perp$. Conversely, if $c \notin \mathrm{Fin}_C$ there is a vector $d \in (\mathrm{Fin}_C)^\perp$ such that $\langle c \mid d\rangle \neq 0$. But $(\mathrm{Fin}_C)^\perp = C^\perp \subset D \cap (-D)$, hence $c \notin (D \cap (-D))^\perp$. □

In other terms, the domain of C is the orthogonal of the null space of $\sim C$.

Definition 10
The domain Fin_C *is naturally equipped with a semi-norm* $\| \cdot \|_C$ *and a preorder* $\preccurlyeq_C$ *:*

$$\begin{array}{rcl} \|x\|_C & = & \sup\{|\langle x \mid d\rangle| \; ; \; d \in D\} \\ x \preccurlyeq_C y & \Leftrightarrow & \forall d \in D \;\; \langle x \mid d\rangle \leq \langle y \mid d\rangle \end{array}$$

Let $\cong_C$ *be the equivalence associated with* $\preccurlyeq_C$.

Proposition 9
The zero space $\mathbf{0}_C$ *of* $\| \cdot \|_C$ *is identical to the zero class modulo* $\cong_C$.

Proof : Obvious. □

In particular, $\mathrm{Fin}_C/\mathbf{0}_C$ is a partially ordered Banach space.

Proposition 10

(i) C^+ *is the set of positive elements modulo* $\preccurlyeq_C$.

(ii) $\mathbf{0}_C = C^+ \cap (-C^+) = C \cap (-C)$.

(iii) *The unit ball w.r.t.* $\| \cdot \|_C$ *is* $(C - C^+) \cap (C^+ - C)$.

Proof : (i) and (iii) come respectively from the cases "$b \notin C^+$" and "$b \notin C - C^+$" in the proof of theorem 4. (ii) is immediate. □

The next properties are more or less reformulations of what we already established.

- *(i)* The partial order $\preccurlyeq_C$ is continuous w.r.t. $\|.\|_C$: if $x_n \preccurlyeq_C y_n$ and $(x_n), (y_n)$ are Cauchy sequences w.r.t. $\| \cdot \|_C$ with limits x, y, then $x \preccurlyeq_C y$.
- *(ii)* If $0 \preccurlyeq_C x \preccurlyeq_C y$, then $\|x\|_C \leq \|y\|_C$.
- *(iii)* If $x \in \mathrm{Fin}_C$, then there exists $y, z \succcurlyeq_C 0$ such that $x = y - z$ and $\|y\| \leq \|x\|$.

Now what is the relation between norm and order w.r.t. C and norm and order w.r.t. $\sim C$? The question is not to establish any new result, everything has been said, but to look for symmetries $C/\sim C$. We consider successively : equivalence, positivity, semi-norm.

Equivalence

$$x \cong_C y \quad \Leftrightarrow \quad \forall x', y'(x' \cong_{\sim C} y' \Rightarrow \langle x \mid y \rangle = \langle x' \mid y' \rangle) \tag{24}$$

The introduction of the domain Fin_C, i.e., the fact of considering a *partial* (non-reflexive) equivalence relation (PER) is responsible for this symmetrical formulation.

Positivity

$$x \in C^+ \quad \Leftrightarrow \quad \forall y(y \in (\sim C)^+ \Rightarrow \langle x \mid y \rangle \geq 0) \tag{25}$$

The relation $\preccurlyeq_C$ is a preorder on the domain Fin_C. I don't know how to call a transitive relation enjoying *weak reflexivity* :

$$x \preccurlyeq y \Rightarrow x \preccurlyeq x \wedge y \preccurlyeq y \tag{26}$$

"partial preorder" conflicts with the use of "partial" in "partial order". I therefore propose to call it a "POR" (like we say "a PER").
The next result generalises the familiar decomposition of a hermitian as a difference $u = u^+ - u^-$ of two positive hermitians, see the default choices in section 3.6.

Theorem 5
Given $x \in \mathbb{E}$ there are unique $x^+ \in C^+$ and $x^- \in (\sim C)^+$ such that $x = x^+ - x^-$ and $\langle x^+ \mid x^- \rangle = 0$.

Proof : Let x^+ be the projection of x on the convex set C, and let $x^- := x - x^+$. It is well-known that x^- is the unique y such that $\langle y \mid x-y\rangle \geq \langle y \mid z\rangle$ for all $z \in C$. This last condition is easily shown to be equivalent to $y \in (\sim C)^+$ and $\langle y \mid x-y\rangle = 0$. ☐

Semi-norm

$$\|x\|_C = \inf\{\lambda \,;\, \forall y \in (\sim C)^+ \quad |\langle x \mid y\rangle| \leq \lambda \|y\|_{\sim C}\} \tag{27}$$

It is not the case that $|\langle x \mid y\rangle| \leq \|x\|_C \cdot \|y\|_{\sim C}$ for all $x \in \mathrm{Fin}_C$, $y \in \mathrm{Fin}_{\sim C}$. I am not sure that one should spend too much time on this, since the choice of the norm makes sense for us only for *positive* elements, as a way of defining *coherence.*

Proposition 11
If $C \subset D$, then :

$$\begin{array}{rcl} \mathrm{Fin}_C & \subset & \mathrm{Fin}_D \\ \preccurlyeq_C & \subset & \preccurlyeq_D \\ \cong_C & \subset & \cong_D \\ \|\cdot\|_C & \geq & \|\cdot\|_D \end{array}$$

The last inequality can be understood by extending $\|\cdot\|_C$ into a *total* function with values in $[0, +\infty]$.

3.6 Quantum coherent spaces

Let $\mathbb{X}$ be a finite-dimensional (complex) Hilbert space ; let $\mathbb{E} = \mathcal{H}(\mathbb{X})$ be the set of *hermitian* (self-adjoint) operators on $\mathbb{X}$. $\mathbb{E}$ is a real vector space (whose dimension is the square of the dimension of $\mathbb{X}$) naturally endowed with the scalar product

$$\langle u \mid v\rangle := \mathrm{tr}(uv) \tag{28}$$

which makes it an Euclidian space : $\mathrm{tr}(uv) = \mathrm{tr}(vu) = \overline{\mathrm{tr}(uv)}$, $\mathrm{tr}(u^2) > 0$ for $u \neq 0$. Two hermitians are said to be *polar* when $0 \leq \langle u \mid v\rangle \leq 1$.

Definition 11
A quantum coherent space *(QCS) with carrier $\mathbb{X}$ is a subset of $X \subset \mathcal{H}(\mathbb{X})$ equal to its bipolar.*

Theorem 4 yields a characterisation of QCS. Some default choices are given by :

Example 3

Negative default : *N consists of all positive hermitians of norm ≤ 1. N^+ therefore consists in all positive hermitians ; on N^+, $\|\cdot\|_N$ coincides with the usual (supremum) norm $\|\cdot\|_\infty$.*

Positive default : *P consists of all positive hermitians of trace ≤ 1 . P^+ therefore consists in all positive hermitians ; on P^+, $\|\cdot\|_P$ coincides with the usual trace norm $\|u\|_1 = \mathrm{tr}(\sqrt{uu^*})$.*

Hilbert-Schmidt default : *H consists of all positive hermitians of Hilbert-Schmidt norm less than 1. H^+ therefore consists in all positive hermitians ; on H^+, $\|\cdot\|_H$ coincides with the usual Hilbert-Schmidt norm $\sqrt{\mathrm{tr}(uu^*)}$. This choice is self-dual : $\sim H = H$.*

In fact, $P = \sim N$; one basically uses $|\mathrm{tr}(uv)| \leq \|u\|_\infty \cdot \|v\|_1$, and, for $u, v \geq 0$, $\quad \mathrm{tr}(uv) = \mathrm{tr}(\sqrt{u}v\sqrt{u}) \geq 0$ and $\mathrm{tr}(uxx^*) = \langle u(x) \mid x\rangle$.

4 Additives

4.1 Basics of quantum physics

Let us recall a few basics of quantum mechanics ; we stay in finite dimension to avoid technical problems.

(i) The state of a system is represented by a *vawe function*, i.e., a vector x of norm 1 in some Hilbert space $\mathbb{X}$.

(ii) A *measurement* is a hermitian operator Φ on $\mathbb{X}$. To say that the *value* of x w.r.t. Φ is λ is the same as saying that $\Phi(x) = \lambda x$. This means that, under normal conditions, there is no value at all. Moreover, if Φ, Ψ do not commute, they are likely to have no common eigenvector, so x cannot have a value w.r.t. both of them, as in the famous *uncertainty principle.* For instance the Pauli matrices (see *infra*) which measure the spin along the axes $\vec{X}, \vec{Y}, \vec{Z}$, do not commute : if the spin is $+1/2$ along the axis $\vec{Z}$, then it is completely undetermined along $\vec{X}$.

(iii) The process of measurement is a Procustus's bed, it forces the system to "have a value". This means, that, after a measurement, the wave function x is replaced with an eigenvector x' of Φ. This process is non-deterministic : in fact, if $\mathbb{X}$ is split as the direct sum of the eigenspaces of Φ : $\mathbb{X} = \bigoplus_\lambda \mathbb{X}_\lambda$, so that $x = \bigoplus_\lambda x_\lambda$, then x' is one of the components x_λ, up to renormalisation (multiplication by $1/\|x_\lambda\|$), and the

probability of the transition $x \rightsquigarrow x_\lambda/\|x_\lambda\|$ is $\|x_\lambda\|^2$. This process is known as the *reduction of the wave packet*, *reduction* for short.

(iv) In this pattern, wave functions make sense up to multiplication by any element of the unit circle. Typically, when we deal with the *spin* of an electron, which is nothing but the quantum analogue of a boolean, a rotation of 2π will replace x with $-x$, without any significant consequence.

(v) *Density matrices* have been introduced by von Neumann ; they take care of the scalar indetermination of wave functions, they also take care of the probabilistic aspect of measurement. A *density operator* is a positive hermitian of trace 1. Density matrices form a compact convex set, whose extremal points are operators of the form xx^*, where x is a vector of norm 1, i.e., a wave function, uniquely determined up to multiplication by a scalar of modulus 1. When one performs a measurement, xx^* is replaced with $\sum_\lambda x_\lambda {x_\lambda}^*$: this density operator is a "mixture", a convex combination of extremal points $x_\lambda {x_\lambda}^*/\|x_\lambda\|^2$, with coefficients $\|x_\lambda\|^2$ which correspond to the respective probabilities of each transition.

(vi) One can iterate measurements, this means, apply this process to an arbitrary density operator, not necessarily extremal. Concretely, this means that we write our density matrix u as a "matrix" $(u_{\lambda\mu})$ w.r.t. the decomposition $\mathbb{X} = \bigoplus_\lambda \mathbb{X}_\lambda$ $(u_{\lambda\mu} \in \mathcal{L}(\mathbb{X}_\mu, \mathbb{X}_\lambda))$, then the reduction of the wave packet consists in annihilating the non-diagonal "coefficients" $u_{\lambda\mu}$: after the measurement, the density matrix becomes $v = (u_{\lambda\mu})$, with $v_{\lambda\lambda} = u_\lambda\lambda, v_{\lambda\mu} = 0$ for $\lambda \neq \mu$.

(vii) The measurement process is irreversible : if $u \rightsquigarrow v$ through measurement, then $\mathrm{tr}(v^2) \leq \mathrm{tr}(u^2)$, i.e., the Hilbert-Schmidt norm decreases[7]. If $\mathbb{X}$ is of dimension n, then the HS norm can vary between 1 (extremal point xx^*) and $1/\sqrt{n}$, which corresponds to $1/n \cdot I$, the "tepid mixture", which conveys no information at all.

4.2 Quantum booleans

4.2.1 Commutative booleans

With start with 2×2 *matrices.* As long as traditional logic is concerned, there is little to say :

(i) The booleans `true,false` are naturally represented by $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

(ii) It is natural to think that a diagonal matrix $\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}$, with $\lambda + \mu = 1$, $\lambda, \mu \geq 0$ represents a probabilistic boolean.

But, as soon as one "forgets the diagonal", i.e., when one considers "booleans of arbitrary basis", then the three —nay the four— dimensions of space come into the picture.

4.2.2 Space-time

Any hermitian can be written $h = 1/2 \begin{bmatrix}t+z & x-iy\\ x+iy & t-z\end{bmatrix}$, i.e., $t.s_0 + x.s_1 + y.s_2 + z.s_3$, where t, x, y, z are real and the s_i are the *Pauli matrices* $1/2 \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $1/2 \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $1/2 \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ $1/2 \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Observe that *time* t is nothing but the *trace*, $t = \mathrm{tr}(h)$. As to the determinant, we get $4\mathrm{det}(h) = (t^2 - (x^2+y^2+z^2))$, the square of the pseudo-metrics. Remark that $\mathrm{tr}((t.s_0+x.s_1+y.s_2+z.s_3)(t'.s_0+x'.s_1+y'.s_2+z'.s_3)) = tt'+xx'+yy'+zz'$. For $1 \leq i \neq j \leq 3$, we have the anti-commutations $s_i.s_j + s_j.s_i = 0$.
In order to characterise *positive hermitians*, remember that, modulo a unitary transformation, $uhu^* = \begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}$, with $\lambda, \mu \in \mathbb{R}$, so that h is positive iff $\lambda, \mu \geq 0$. In other terms, the condition $\mathrm{det}(h) \geq 0$ (vectors in position "time") characterises hermitiens which are either positive or negative. Positive hermitians correspond to the further requirement $\mathrm{tr}(h) \geq 0$, i.e., to the "cone of future".
The most general transformation preserving positive hermitians is of the form $h \rightsquigarrow uhu^*$, with $\mathrm{det}(u) = 1$, i.e., $u \in \mathtt{SL}(2)$: such transformations correspond to the familiar positive Lorenz group, which is the group of linear transformations preserving the pseudo-metrics and the future. By the way, observe that the inverse of* $u \in \mathtt{SL}(2)$ is given by :

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix} \tag{29}$$

Therefore, inversion can be extended into an involutive anti-automorphism of the C^*-algebra $\mathcal{M}_2(\mathbb{C})$ of 2×2 matrices. This anti-automorphism acts on space-time by negating the spacial coordinates.
The positive Lorenz admits as a subgroup the group `SO(3)` of *rotations*,

which modify only space : they correspond to trace-preserving transformations, those who are induced by *unitaries*. In other terms, `SO(3)` admits a double covering by `SU(2)`, the group of unitary transformations of determinant 1, whose general form is $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, with $a\bar{a} + b\bar{b} = 1$. The rotations of axes $\vec{X}, \vec{Y}, \vec{Z}$ and angle θ are induced by the unitaries $e^{i\theta s_k}$, i.e., $\begin{bmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$ $\begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{bmatrix}$ $\begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$ respectively. Remark the "heresy" consisting in dividing an angle by 2, an operation with two solutions... This is why one speaks of a double covering ; this is also why a rotation of angle 2π acts on a spin (seen as a wave function) by multiplying by -1.

4.2.3 Quantum booleans

"Classical" booleans correspond to projections on two 1-dimensional subspaces which are distinguished by the matricial representation. A *quantum* boolean will therefore be a subspace of dimension 1. By the way, remark that this definition refuses any differentiation between true and false : if the space E is a quantum boolean, its negation is $E^{\perp}$, period. By the way, remark that, due to problems of commutation, it will be impossible to construct convincing binary connectives. It remains to determine the subspaces of dimension 1, i.e., the matrices of orthogonal projections of rank 1. Those are the hermitian matrices of trace 1 and determinant 0, i.e., the points of space-time $t.s_0 + x.s_1 + y.s_2 + z.s_3$, with $t = 1, x^2 + y^2 + z^2 = 1$, which are therefore in $1-1$ correspondence with the sphere S^2. What we just explained is the natural way to speak of a quantum boolean, which also known to physicists as the *spin* of an electron.

4.2.4 Probabilistic quantum booleans

Probabilistic quantum booleans (PQB) are just convex combinations of quantum booleans, i.e., "density matrices", positive hermitians of trace 1. Any PQB can be diagonalised in an orthonormal basis. In which respect is this unique ?

(i) The PQB $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ is diagonal in all bases. This is the extreme form of non-unicity.

(ii) Apart from this case, our boolean can be written $\lambda b + (1-\lambda)c$, where b, c

are quantum booleans and $0 \leq \lambda \leq 1$. λ, b, c are uniquely determined if we require $0 \leq \lambda < 1/2$.

The reduction of the wave packet occurs when we want to measure a boolean, this corresponds to the measurement of a spin in physics. First we must specify an orthonormal basis, and write operators as matrices w.r.t. this base. Say that our PQB corresponds to the matrix $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$, then, after measurement, it becomes $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, i.e., true with probability a, false with probability $c = 1 - a$.

4.2.5 Negation

Specifying an orthonormal basis consists in chosing two orthogonal subspaces of dimension 1, i.e., two quantum booleans π and $1 - \pi$, whose four-dimensional coordinates are therefore $(1, x, y, z)$ and $(1, -x, -y, -z)$. The two vectors $\vec{A} = (x, y, z)$ and $-\vec{A}$ correspond to two opposite directions on the same three-dimensional axis (spin up, spin down). The symmetry w.r.t. origin comes from the anti-automorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ of the C^*-algebra $\mathcal{M}_2(\mathbb{C})$ of 2×2 matrices. This transformation corresponds to *negation*. It must be observed that, since symmetry w.r.t. the origin is of determinant -1, it is not in `SO(3)`, and therefore it is not induced by an element of `SU(2)`.

4.2.6 Binary boolean connectives

Whereas negation does not need reduction, binary boolean connectives will badly need it ; there are two reasons for that.

(i) We cannot combine non-commuting 1-dimensional projections in a way that will produce another projection.

(ii) Common sense tells us that, if we cannot distinguish between true and false, then we cannot distinguish between conjunction and disjunction.

Hence binary connectives will be probabilistic : they yield a PQB even when the inputs are "pure" quantum booleans. Moreover, they depend on the choice of a basis, and an order of evaluation ; I give an example : $\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \vee \begin{bmatrix} a' & b' \\ \bar{b}' & c \end{bmatrix} := \begin{bmatrix} a + ca' & cb' \\ c\bar{b} & cc' \end{bmatrix}$. The first argument is "reduced" in the canonical base : true with probility a, in which case the answer is $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, false with probility c, in which case the answer is $\begin{bmatrix}a' & b'\\ \bar{b}' & c\end{bmatrix}$. There is a symmetrical choice which reduces the second argument. But only a real Jivaro will choose the third possibility, which reduces *both* arguments, yielding $\begin{bmatrix}a+ca' & 0\\0 & cc'\end{bmatrix}$, which is in fact symmetrical, since $a+ca' = a'+ca = a+a'-aa'$.

4.3 Quantum and additives

4.3.1 Basics

Definition 12
If X, Y are QCS with respective carriers $\mathbb{X}, \mathbb{Y}$, one defines the additive combinations $X \oplus Y$ and $X \,\&\, Y$, as QCS of carrier $\mathbb{X} \oplus \mathbb{Y}$.

$$\begin{aligned} X \oplus Y &= \{\lambda u \oplus (1-\lambda)v \ ; \ u \in X, v \in Y, 0 \leq \lambda \leq 1\} \\ X \,\&\, Y &= \{w \ ; \ \mathbb{X}w\mathbb{X} \in X, \mathbb{Y}w\mathbb{Y} \in Y\} \end{aligned}$$

As usual, we have identified the subspaces $\mathbb{X}$ and $\mathbb{Y}$ with the associated orthogonal projections.

Proposition 12
$\oplus$ and $\&$ are swapped by negation.

Proof : Essentially because $\langle u \oplus v \mid u' \oplus v' \rangle = \langle u \mid u' \rangle + \langle v \mid v' \rangle$. □

Observe that $\| \cdot \|_{X \oplus Y}$ and $\| \cdot \|_{X \& Y}$ are not norms. This is because this definition mistreats all hermitians which are not of the form $u \oplus v$. W.r.t. an obvious matricial notation, every hermitian on $\mathbb{X} \oplus \mathbb{Y}$ can be written $\begin{pmatrix} u & w \\ w^* & v \end{pmatrix}$, with, u, v hermitian. If $w \neq 0$, then this operator has infinite norm in $X \oplus Y$. *A contrario*, its norm w.r.t. $X \,\&\, Y$ does not depend on w : the null space $\mathbf{0}_{X \& Y}$ contains all $\begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}$.

4.3.2 Dimension 2

If $\mathbb{X}$ is of dimension 1, then $\mathcal{H}(\mathbb{X})$ is of dimension 1 (isomorphic to $\mathbb{R}$) and the three defaults of example 3 coincide, and yield the same QCS, noted $\mathbf{1}$, which corresponds to the segment $[0,1]$ of $\mathbb{R}$. The ordering is the usual ordering, and the norm the usual absolute value.
In dimension 2, $\mathcal{H}(\mathbb{X})$ has dimension 4, and there are many choices.

Spin : the positive default. The elements of **Spin** are positive hermitians of trace at most 1. They are not quite PCB, since a PCB is of trace 1, they are sort of "partial PCB". Concretely, if we measure an element $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$, it will yield "true" with probability a, "false" with probability c, and nothing with probability $1-a-c$. This "nothing" is natural from the computational viewpoint : if we assume that the measurement is done through a computing device, then we are likely to wait before getting our probabilistic answer "true" or "false". "Nothing" corresponds to the case of a computing loop, i.e., when we wait too long.

∼Spin : the negative default. The elements of $\sim$ **Spin** are positive hermitians of (usual) norm at most 1. They should be understood as "anti"-booleans.

Bool : the "Plus" of two copies of $\mathbf{1}$. The space $\mathbf{1}\oplus\mathbf{1}$ consists of all diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ such that $0 \leq a, c \leq a+c \leq 1$. This QCS is a subset, a "subtype" of **Spin**. It has a well-defined notion of truth and falsity.

∼Bool : the negation of the former, i.e., $\mathbf{1}\ \&\ \mathbf{1}$. It consists in all matrices $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ such that $0 \leq a, c \leq 1$.

Now observe that our construction of **Bool** depends on the choice of a 1-dimensional subspace (corresponding to "true"). This means that, given any vector $\vec{A} \in S^2$, there is a QCS made of "booleans of axis $\vec{A}$", noted $\mathbf{Bool}_{\vec{A}}$.

Proposition 13
$\mathbf{Spin} = \bigcup_{\vec{A}\in S^2} \mathbf{Bool}_{\vec{A}}$.

Proof : Obviously $\mathbf{Bool}_{\vec{A}} \subset \mathbf{Spin}$. Conversely, if $h \in \mathbf{Spin}$, it can be put in diagonal form $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, with $0 \leq a, c \leq a+c \leq 1$, w.r.t. a certain basis $\mathbf{e}, \mathbf{f}$. If $\vec{A}$ is the point of S^2 corresponding to $\mathbf{e}$, then $h \in \mathbf{Bool}_{\vec{A}}$. ☐

Corollary 13.1
$\sim\mathbf{Spin} = \bigcap_{\vec{A}\in S^2} \sim\mathbf{Bool}_{\vec{A}}$.

4.3.3 Reduction : a discussion

In the next section, we shall deal with multiplicatives and linear implication. In particular, we shall be able to transform a boolean $h \in \mathbf{Spin}$ into something else by using an element of the QCS $\mathbf{Spin} \multimap \ldots$, then transform the result by means of another implication... Some of these transformations will behave like negation (wave-like) others will use reduction. We try now to understand to which extent reduction is subjective. For this, we make an impossible thing, we assume that the process of transformation is over, i.e., that in this sequence of successive implications, we have succeeded in "closing the system". This means that there is an ultimate implication with values in $\mathbf{1}$. If I compose all my implications, I eventually discover that a sequence of transformation, eventually "closing the system" is exactly an anti-boolean $k \in \sim\mathbf{Spin}$. The resulting output is objective : $\langle h \mid k\rangle = \mathrm{tr}(hk)$. But the choice of k (the transformations, observations made on h) is highly subjective, we are biased, we are "on the side of k). If we are on the side of k, then put k in diagonal form w.r.t. a basis $\mathbf{e}, \mathbf{f}$. Then $h = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}, k = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$, so that $\langle h \mid k\rangle = a\alpha + c\gamma$. If $h' = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, then $\langle h \mid k\rangle = \langle h' \mid k\rangle$, i.e., it is as if h had been reduced.
It may be the case that we know that f is a boolean in a certain base (e.g., if f is the result of a measurement). Then we can select this base, in which case $h = \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix}, k = \begin{pmatrix} \alpha' & \beta' \\ \bar{\beta}' & \gamma' \end{pmatrix}$, and we can write $\langle h \mid k\rangle = a'\alpha' + c'\gamma'$. In that case, we can "reduce" the observer k into $k' = \begin{pmatrix} \alpha' & 0 \\ 0 & \gamma' \end{pmatrix}$ so that $\langle h \mid k\rangle = \langle h \mid k'\rangle$. This shows the extreme subjectivity of reduction.

5 Multiplicatives

5.1 Linear functionals

Theorem 6
Let $\mathbb{X}, \mathbb{Y}$ be finite dimensional Hilbert space. Then $\mathcal{L}(\mathcal{L}(\mathbb{X}), \mathcal{L}(\mathbb{Y})) \simeq \mathcal{L}(\mathbb{X} \otimes \mathbb{Y})$.

Proof : The complex vector space $\mathcal{L}(\mathbb{X})$ is generated by rank 1 endomorphisms $xw^* : xw^*(y) = \langle y \mid z\rangle x$. If $\varphi \in \mathcal{L}(\mathcal{L}(\mathbb{X}), \mathcal{L}(\mathbb{Y})$, define $\Phi \in \mathcal{L}(\mathbb{X} \otimes Y)$ by

$$\langle \Phi(x \otimes y) \mid w \otimes z\rangle = \langle \varphi(xw^*)(y) \mid z\rangle \tag{30}$$

Conversely, given $\Phi \in \mathcal{L}(\mathbb{X} \otimes Y)$, if $f \in \mathcal{L}(\mathbb{X})$, then one defines $[\Phi]f \in \mathcal{L}(\mathbb{Y})$

by :

$$\langle([\Phi]f)(y) \mid z\rangle = \mathrm{tr}(\Phi \cdot (f \otimes yz^*)) \tag{31}$$

so that $[\Phi]\cdot \in \mathcal{L}(\mathcal{L}(\mathbb{X}), \mathcal{L}(\mathbb{Y}))$. ☐

Corollary 13.2
If $\Phi \in \mathcal{H}(\mathbb{X} \otimes Y)$, if $f \in \mathcal{H}(\mathbb{X})$, then $[\Phi]f \in \mathcal{H}(\mathbb{Y})$. The map $\Phi \rightsquigarrow [\Phi]\cdot$ is a bijection from $\mathcal{H}(\mathbb{X} \times \mathbb{Y})$ onto the set of linear maps from $\mathcal{H}(\mathbb{X})$ to $\mathcal{H}(\mathbb{Y})$.

Proof : An easy computation shows that $[\Phi^*]f^* = ([\Phi]f)^*$, hence a hermitian Φ sends hermitians to hermitians. Conversely, if φ is a linear map from $\mathcal{H}(\mathbb{X})$ to $\mathcal{H}(\mathbb{Y})$, then φ can be uniquely extended into a $\mathbb{C}$-linear map from $\mathcal{L}(\mathbb{X})$ to $\mathcal{L}(Y)$: $\varphi(u) = 1/2(\phi(u + u^*) + i\phi(iu^* - iu))$. Now the $\mathbb{C}$-linear maps obtained in this way are *hermitian*, i.e., $\varphi(f^*) = \varphi(f)^*$, and they are in 1-1 correspondence with hermitians of $\mathcal{H}(\mathbb{X} \otimes Y)$. ☐

The essential property of $[\phi]\cdot$ is summarised by the equation

$$\mathrm{tr}(([\Phi]f) \cdot g) = \mathrm{tr}(\Phi \cdot (f \otimes g)) \tag{32}$$

Example 4
If $\sigma_{\mathbb{X}} \in \mathcal{H}(\mathbb{X} \otimes \mathbb{X})$ is such that $\sigma(x \otimes y) = y \otimes x$ (the "flip"), then $\langle[\sigma](xw^*)(y) \mid z\rangle = \langle\sigma x \otimes y \mid w \otimes z\rangle = \langle y \otimes x \mid w \otimes z\rangle = \langle y \mid w\rangle\langle x \mid z\rangle = \langle(xw^*)(y) \mid z\rangle$. Hence $[\sigma]xw^* = xw^*$ and by linearity $[\sigma]f = f$.

Example 5
More generally, let u be any map from $\mathbb{X}$ to $\mathbb{Y}$. Then $u \otimes u^*$ maps $\mathbb{X} \otimes \mathbb{Y}$ into $\mathbb{Y} \otimes \mathbb{X}$, and if $\sigma_{\mathbb{X}\mathbb{Y}}$ is the "flip" from $\mathbb{Y} \otimes \mathbb{X}$ to $\mathbb{X} \otimes \mathbb{Y}$, then $U = \sigma \cdot u \otimes u^* \in \mathcal{H}(\mathbb{X} \otimes \mathbb{Y})$. It is immediate that $[U]f = ufu^*$.

Example 6
Let $1_{\mathbb{X}} = E + F$ be a decomposition of the identity as a sum of orthogonal projections (subspaces). Then $R = \sigma(E \otimes E + F \otimes F)$ acts as follows : $[R]f = EfE + FfF$. R is a typical reduction operation, it chops off the "non-diagonal" portions EfF and FfE of f.

One can wonder what is the status of the identity map of $\mathbb{X} \otimes \mathbb{Y}$. An easy computation shows that $[1_{\mathbb{X}\otimes\mathbb{Y}}](u) = \mathrm{tr}(u) \cdot 1_{\mathbb{Y}}$. Not very exciting… But thisquelque part will help us with our last example :

Example 7
If $\mathbb{X}$ is of dimension 2, then $[1_{\mathbb{X}\otimes\mathbb{X}} - \sigma_{\mathbb{X}}]\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} c & -b \\ -\bar{b} & a \end{pmatrix}$, i.e., acts like negation. *Observe that $1_{\mathbb{X}\otimes\mathbb{X}} - \sigma_{\mathbb{X}} = 2\pi$, where π is the orthogonal*

projection corresponding to the antisymmetric (one-dimensional) subspace of $\mathbb{X} \otimes \mathbb{X}$, i.e., the space of vectors $x \otimes y - y \otimes x$.

5.2 Connectives

Definition 13

Let X, Y be QCS with respective carriers $\mathbb{X}, \mathbb{Y}$. We define the QCS $X \multimap Y$, with carrier $\mathbb{X} \otimes \mathbb{Y}$, as the set of all Φ sending $\mathbb{X}$ to $\mathbb{Y}$:

$$X \multimap Y \quad = \quad \{\Phi \; ; \; \forall f \in X \;\; [\Phi]f \in Y\} \tag{33}$$

$X \multimap Y$ could as well be defined by

$$X \multimap Y \quad = \quad \{\Phi \; ; \; \forall g \in {\sim} Y \;\; g[\Phi] \in {\sim} X\} \tag{34}$$

and also as ${\sim}\{f \otimes g; f \in X, g \in {\sim} Y\}$. This last expression shows that $X \multimap Y$ is a QCM. From this we can define $X \parr Y = {\sim} X \multimap Y$ and $X \otimes Y = {\sim}{\sim}\{f \otimes g; f \in X, g \in Y\}$. As usual, $\parr$ is commutative, associative, and distributive over & (all this up to isomorphism).

As usual, "Times" is more difficult to access than "Par". By equation (25) (and Hahn-Banach) one can characterise the "positive" cone of a "Times", as the closure of the set of finite sums $\sum_i f_i \otimes g_i$, $f_i, g_i \geq 0$[8]. In the same way, (27) can be used to determine the semi-norm associated with a "Times".

Remark 1

It is important to observe that multiplicatives force a departure from the standard ordering of hermitians. For instance, assume that X, Y have been equipped with the positive defaults, e.g., $X = Y =$ **Spin**. *Then $X \multimap Y$ will declare as positive any hermitian sending positive hermitians to positive hermitians. The most typical example is the* flip *σ which behaves like the identity map. But σ is a proper symmetry, not a positive hermitian. So $X \multimap Y$ is more liberal as to positivity than expected. This means that, dually, $X \otimes Y$ is more restrictive. In fact, the positive cone of $X \otimes Y$ is the closure of the set of finite sums $\sum_i f_i \otimes g_i$, $f_i, g_i \geq 0$. Most positive hermitians on $\mathbb{X} \otimes \mathbb{Y}$ cannot be obtained in this way : take any orthogonal projection zz^*, where z is not a pure tensor !*

5.3 η-expansion and reduction

The question "is a function a graph ?" is traditional in logic, and quite scholastic. It is such a long time that people exchange the same arguments ; do they actually believe in what they say ? There is peculiar form of this question, known as “η-conversion, and limited to the sole identity function. Given a logically compound formula F, then the identity function admits two alternative descriptions, as a proof of $C \multimap C$:

Generic : since C is identical to C, the *identity axiom* maps C into C.

η-expanded : decompose C into components, $A, B, \ldots$, and recompose the identity functions of $A, B, \ldots$, in order to produce an identity function of C.

The two processes are identified by all *honest* interpretations, i.e., interpretations which are not contrived to make a difference between them. This is why, in my own *ludics*, [9], everything was “η-expanded”, i.e., the identity was not primitive.
We shall show that η-expansion is wrong, by differentiating the identity from its η-expansion in the case $C = A \oplus B^9$. For simplicity, let us assume that A, B have both carriers of dimension 1. Our two identities respectively correspond to :

The flip : the generic identity map of a space $\mathbb{X}$ of dimension 2. This map writes as $\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ in any base $\mathbf{e} \otimes \mathbf{e}, \mathbf{e} \otimes \mathbf{f}, \mathbf{f} \otimes \mathbf{e}, \mathbf{f} \otimes \mathbf{e}$ of $\mathbb{X} \otimes \mathbb{X}$.

The η-expanded flip : it corresponds to putting together two identities. W.r.t. a specific base (corresponding to the decomposition of C as a direct sum), it writes $\iota = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

These two maps are clearly distinct : $[\sigma] \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$, it is the real identity. On the other hand, $[\iota] \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ is a Procustus's identity. It behaves as the identity w.r.t. matrices which already have the right logical form $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, and those who don't follow the logical rule, it chops off their anti-diagonal coefficients. Of course, if we remember our basics, ι is quite the reduction of the wave packet, corresponding to the measurement

of *spin* along the vertical axis $\vec{Z}$.
In logic, only the identity can be η-expanded, but this is an accident. For instance the negation $\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is such that $[\nu]\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} c & -b \\ -\bar{b} & a \end{pmatrix}$ w.r.t. a given base can be η-expanded into $\nu' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; obviously $[\nu']\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & a \end{pmatrix}$: ν' corresponds to a measurement of the spin along the axis $\vec{Z}$ and a subsequent inversion.
To come back to the original question about functions and graphs. In the "commutative" world, every function is bound to be a graph. This is because everything is diagonal in a fixed basis. When the distinguished basis disappear, the "atoms" disappear as well. η-expansion corresponds to the choice of a set of atoms (a basis), the decomposition of a function along this basis, and its recomposition. This process is violently incorrect in a non-commutative setting.

5.4 Still to be done

The main challenge is the extension to infinite dimension :

(i) First, the approach is not fully Augustinian, since the carriers $\mathbb{X}, \mathbb{Y}$ are "taken from a hat". It would be nicer to fix once for all a separable Hilbert space.

(ii) Second, the imperfect (infinite) part of logic needs to be studied too. It is to be remarked that the exponential $!A$ "forever A" is much bosonic in spirit. In general the question of a possible logical status for the two types of quantum symmetry (fermionic, bosonic) is much exciting.

However, this stumbles on serious problems.

(i) Köthe spaces, as used by Ehrhard, see section 3.1, are perfect as an infinite *commutative* Augustinian explanation of logic. One can fix once for all a denumerable index set $\mathbb{I}$ and define polarity by :

$$f \overset{|}{\sim} g \quad \Leftrightarrow \quad |\sum_{i \in \mathbb{I}} f(i) \cdot g(i)| \leq 1 \tag{35}$$

But this approach does not allow significant changes of basis, and is inappropriate for quantum.

(ii) Finite-dimensional Hilbert spaces give rise to type I_n *factors*, i.e., "connected" von Neumann algebras. The most trivial generalisation is a type I_∞ factor, i.e., the space $\mathcal{B}(\mathbb{H})$ of *bounded operators* on an infinite-dimensional Hilbert space. The main problem is that such an algebra is *semi-finite*, i.e., trace makes sense, as an element of $[0, +\infty]$, only for *positive* operators. But we badly need equations like $[\sigma_{\mathbb{X}\otimes\mathbb{X}}]\sigma_{\mathbb{X}} = \sigma_{\mathbb{X}}$, which has strictly no meaning from this viewpoint.

(iii) Another direction would be type II_1 factors, typically the famous matricial factor, which harbours a (unique) finite trace. But $\mathrm{tr}(\sigma \cdot 1 \otimes 1) = \mathrm{tr}(\sigma) = 0 \neq \mathrm{tr}(1 \cdot 1) = 1$. The reason for this vanishing of σ is the same as the reason of the vanishing of Δ in section 3.2.

What is most likely to happen is the use of a matricial factor of type II_1 together with the replacement of trace with determinant, $\det(1-\mathrm{uv})$, instead of $\mathrm{tr}(uv)$. But this involves *geometry of interaction*, see [6], and this is quite another story.

5.5 Relation to quantum computing

Although it is not my primary interest, the relation to quantum computing should be considered. It would be interesting to revisit Selinger's language for quantum computation [11] in the spirit of QCS. However, the use of loops in the style of geometry of interaction may suggest that determinant might be more appropriate. Perhaps more appropriate (because explicitely based on linear logic) is the "quantum lambda-calculus" recently proposed by van Tonder [12].

NON SI NON LA

Notes

[1] Augustine proposed to define Good end Evil, not as absolute manicheist essences, but through their interaction.

[2] Witness the failure of all attempts at axiomatising subtyping. Such a thing shouldn't even be tried, since an axiomatisation keeps a distance between object and subject, hence treats objects up to isomorphism and cannot make sense of an inclusion.

[3] The question of an Augustinian approach to related notions such as *hypercoherences*, [3], is still open.

[4] And $!X$ is a sort of symmetric (co)-algebra, much *bosonic* in spirit... but this is beyond the scope of this paper.

[5] Cartesian product, like "Par", is associative only up to isomorphism.

[6] The formula defines in fact what I call $\mathtt{Fin}_C$, see definition 9.

[7] The reduced hermitian is not smaller : the difference has null trace, and can hardly be positive.

[8] This is obviously related to *separable mixed states*, see, e.g., [10].

[9] But η stays correct in the case of a "Times".

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