From foundations to ludics

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Ludics [1] is a novel approach to logic —especially proof-theory. The present introduction emphasises foundational issues.

1 All Quiet on the Western Front

For ages, not a single *disturbing* idea in the area of « foundations » : the discussion is sort of ossified —as if everything had been said, as if all notions had taken their definite place, in a big cemetery of ideas. One can still refresh the flowers or regild the stone, e.g., prove technicalities, sometimes non-trivial ; but the real debate is still : this paper begins with an *autopsy*, the autopsy of the foundational project.

1.1 Realism

Up to say 1900, the realist/dualist approach to science was dominant ; during the last century some domains like physics evolved so as to become completely anti-realist ; but this evolution hardly concerned logic.

1.1.1 Hilbert's legacy

By the turn of the XX$^{\text{th}}$ century mathematics was jeopardised by paradoxes, the most famous of them being due to Russell. Hilbert's reaction was to focus on *consistency*. But the reduction of paradoxes —and therefore of foundations— to solely the *antinomies* is highly questionable[1] : indeed, the

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[1] $\delta\acute{o}\xi\alpha$: dogma, opinion, intuition. . . : a *paradox* need not be a formal contradiction.

typical paradoxical artifacts are *secret sharers*, objects satisfying the formal definitions but far astray from the intended meaning, typically the Peano « curve » which contradicts our perception of *dimension*. Fortunately, topology has been able to show that dimension m is not the same as dimension n... but just for a second, forget this and imagine *consistent* mathematics in which balls in any dimension are homeomorphic : what a disaster ! This exclusive focus on consistency —not to speak of the strategic failure of the Programme— should explain why logic, especially *foundations* lost contact with other sciences during last century.
Indeed Hilbert's Programme is not quite realistic, it is *procedural*, see 1.3 below : Hilbert tried to avoid as much as possible the external *reality*. This was wise ; but he made several mistakes, both technical and methodological :

- We shall see that Hilbert's Programme relies on a duality between proofs of A and proofs of $\neg A$. But the dualising object, the *pivot* of this duality —the proofs of the absurdity— is empty and this leads to a disaster, see 2.1.2, 2.1.3.

- The mere idea of foundations of mathematics *inside* mathematics is questionable, there is an obvious *conflict of interest*, which is not fixed with the controversial idea of *metamathematics*[2]. Indeed these *metamathematics* are just a part of ordinary mathematics —possibly with a hand tied in the back : this remark underlies Gödel's incompleteness. What remains of this is a sort of *diabolus in logica* : the foundational discourse, once embedded in mathematics yields artificial formulas, —which rather look like sophisms : « I am not provable ». Those are still secret sharers, but more difficult to cope with than their topological ancestors : after the curve with no derivative, the sentence with no meaning !

- The current formalist ideology says that mathematics is a pure play on symbols. This underlies notational quarrels : if operation $\mathcal{M}$ distributes over operation $\mathcal{A}$, a formalist may insist on using $\mathcal{M} = \oplus, \mathcal{A} = \otimes$ or something of the like « you know, notations are arbitrary », implicitly denying any special value to distribution. The mistake might lie in a subtle shift from *pure play* to « meaningless » : who told us that a play on symbols is meaningless, has not its own geometry ? By the way, Gentzen —Hilbert's most conspicuous follower— disclosed a structure (sequent calculus) underlying the « play on symbols »... and sequent calculus has its own geometrical structure —this is precisely what ludics is about.

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[2] $\mu\acute{\epsilon}\tau\alpha$ means various things, mainly « besides, after », for instance in *metaphysics*. In logic the expression conveys —together with « intensional »— a magical connotation : this irrationalism is the hidden face of positivism.

The failure of Hilbert's Programme —a reductionist procedural explanation— was felt like the total victory of realism. What remains of Hilbert's spirit in the « official » approach to foundations is a lurking positivism which underlies the pregnancy of this strange animal —the meta— which accounts both for the failure of the programme and the cracks in the realistic building.

1.1.2 Object vs. subject

The current explanation of logic distinguishes between the world (objective) and its representation (subjective), the *object* and the *subject*. Logical realism relies on an opposition between *semantics* (the world) and *syntax* (its representation) : this opposition is expressed through soundness and (in)completeness. The approach is highly problematic :

- It is a bit delicate to say that natural numbers are « the world ». $\{0, 1, 2\}$ does make a small world, but not $\{0, 1, 2, \ldots\}$: « ... » is a pure fantasy, nobody knowing how to interpret it. The only uncontroversial fact is that *we* have efficient ways of manipulating « ... » —typically induction axioms : integers as satisfying « all » inductions, this is what both Dedekind and Peano tried to catch, with different technicalities. Natural numbers and their predicates, the object and the subject, they are in a egg-and-hen relationship, in particular it is difficult to think of one independently of the other.

- Nevertheless the « subjective » part receives no status at all : a plain formal bureaucracy, subject to rather arbitrary choices. For instance certain authors will insist on minimising the number of connectives or the number of rules, e.g., do everything with the connective « equivalent ». The fact that predicate calculus is complete becomes a pure miracle : why should this bureaucracy correspond to something natural ?

- The same holds when one plugs in incompleteness : not everything is provable, but there are weaker or stronger systems. The image is that of a Big Book in which we can eavesdrop : we can see part of it, but only part of it. Then we can classify the keyholes, think of « reverse mathematics », of the « ordinal strength » of theories. Again there is not the slightest explanation for the coexistence of various systems of arithmetic : simply because the « subjective part » gets no autonomous status.

- In fact there is a complete absence of explanation. This is obvious if we look at the Tarskian « definition » of truth « A is true iff A holds ». The question is not to know whether mathematics accepts such a definition but if there is any contents in it... What is disjunction ? Disjunction is disjunction... What is the solution to $x^2 = 3$? It is the set of numbers a such that $a^2 = 3$... The distinction between $\vee$ and a hypothetical meta-$\vee$ is just a way to avoid the problem : you ask for real money but you are paid with meta-money.

- Classical logic is the logic of reality, the realistic logic. The realist paradigm, which is criticisable in the classical case, becomes impossible as soon as we step out of classical logic. An interesting exercise is the building of a *Broccoli* logic : it consists in introducing new connectives, new rules, the worst you can imagine, and then to define everything *à la Tarski.* Miracle of miracles, completeness and soundness still hold : just because the reality has been defined from its representation : typically the semantics is the syntax **in boldface** ; but this nonsense gets a beautiful name, the *free Broccolo.*

- Moreover there is a basic mistake in the idea of semantics-as-reality. Everybody has seen a proof, maybe not a formal one, but at least something that can be transformed into a formal proof by a computer. But nobody has ever seen the tail of a model, models are ideal (usually infinite) objects. Moreover, what is the destiny of a proof ? It is to be combined with other proofs : theorem A can be used as a lemma to yield B. This suggests looking for an *internal* explanation.

1.1.3 The foundational Trinity

Usual « foundations » therefore involve three layers, just like the Christian Trinity :

Semantics : A first, irreducible, layer is realistic : properties are true or false, they refer to some absolute external reality, the integers —you know $0, 1, 2, \ldots$ Think of the Father.

Syntax : Due to incompleteness, a second partner appears : there are stronger and weaker theories, i.e., ways of accessing this absolute truth. This is the Son (a.k.a. *Verb*) in the Foundational Trinity.

Meta : Truth is « defined » via a pleonasm, Tarskian semantics (a.k.a. *vérité de La Palice*) ; later on, this absolute truth turns into a sort of *Animal Farm* where some are truer than others ; even worse, among the truths some are « predicatively correct », some are not, think of the consistency of « predicative arithmetic » : in the Library of Truth, these incorrect truths are relegated to the *inferno* together with the licentious books of

> Marquis de Sade. Some sort of glue —or smoke, is welcome : this is the Meta, which is the Holy Ghost (a.k.a. go-between) of the Trinity.

The Trinity fixes the Tarskian pleonasm by allowing the Father to be no more than the « meta » —the genitor— of the Son, which in turn can be the meta of somebody else : « Turtles all the way down », each turtle sitting on a bigger metaturtle. The various layers are related through (in)completeness and soundness. It is not impossible that this construction eventually collapses —like the epicycles of Ptolemaeus... or simply like the stack of turtles in a famous Japanese riddle. By the way, we shall see in 1.3.1 and 3.4.4 that completeness is better described as an internal property.

1.1.4 The Thief of Baghdad

The ludic programme abolishes the distinction syntax/semantics. At least at the deepest level : practically speaking the distinction may be seen as a polarisation : *my* viewpoint is rather syntax, whereas my *opponent*'s viewpoint is perceived (by me) as semantics. As the name suggests, *ludics* takes its inspiration in the game-theoretic paradigm :

- Formulas = Games
- Proofs = Winning strategies
- Truth = Existence of a winning strategy

This tends to collapse the first two partners of the Trinity —thus denying any role to the meta. But this is an obstinate guy : close the door, most likely he will try the window, here *the rule of the game*. Who tells you that this move is legal or not, who determines the winner... For instance Gödel's *Dialectica* interprets A by something like $\exists x^\sigma \forall y^\tau \varphi(x,y) = 0$. x, y can be seen as sort of strategies in a game, whose rule is given by φ. This involves a splitting in two layers : the players vs. the « referee », i.e., the meta. The same can be said of all modern « game semantics » who —although more clever than the antique *Dialectica*— respect this dichotomy of a rule of the game on top of an interaction : game semantics admits two layers of semantics, the opponent-as-semantics, and the rule-as-semantics, the latter being a sort of meta-semantics ; this schizophrenia is symptomatic of an archaic stage of foundations.
As we shall see in 3.3.3, ludics closes the window too, by allowing the rule to be part of the game, this is the idea of a *game by consensus* : the role of the referee is played by the opponent. This means that the opponent has a lot of losing strategies —called dog's play— whose only effect is to control your moves : if I move in an « incorrect » way, then he is likely to use one of these dog's strategies and produce a *dissensus*, i.e., argue forever ; but if I move

« correctly », the same strategy will be consensual and nicely lose. This is symmetrical : I have my own dog's strategies to control his moves…
The door is closed, the window too, but the cellar is open : the thief can still rely on the myth of the « ambient space ». This is simple, I give an explanation, but my explanation is done in a language (e.g., classical set-theory) which is my meta, one cannot escape one's meta… This argument has been heavily used against any kind of non-realistic foundations, e.g., against intuitionistic logic. People will agree on the distinction between $\exists n$ and $\neg\neg\exists n$, one is effective, the other not, but this is a matter of eavesdropping : in the « ambient » space, the distinction vanishes.

1.1.5 The ambient space

So everything is embedded in a classical « ambient space » : this is the ultimate weapon of the meta. What to say about this sophism ?
A first remark is that alternative foundations cannot be done in alternative mathematics —think of the unfortunate *ultra-intuitionism*. On the other hand, set-theory is flexible enough to harbour various parts of mathematics, for instance operator algebras : but I would not dare to say that set theory is more basic than operator algebra ! In fact, the use of set-theory is the recognition of a *convenience*, by no means of a foundational status.
Let us take a geometrical analogy : in the XIX$^{\text{th}}$ century, non-Euclidian geometries were introduced, the most basic example being that of a sphere *embedded* in Euclidian space. Non-classical logic —intuitionistic or linear— can be likewise eventually reduced, embedded, in classical logic, set-theory… But « embeddable » is not the same as « embedded », i.e., *varieties* make sense by themselves —even if science-fiction finds it convenient to bypass speed limitations by shortcutting through a surrounding Euclidian *hyperspace*. When we embed everything in a classical universe, are we doing something necessary, canonical, meaningful, or are we just using a hygienic convenience ?
Logic is about thought, not about concrete objects. Foundations are impossible *ex nihilo* since they would involve a thought located outside the thought. For similar reasons it is *a priori* impossible to take a photo of the universe since one would need a point of view outside space ; but cosmology made a lot of progress last century, without allowing any room for a « meta-space-time ». This analogy makes us understand that the so-called « meta » could be nothing more than the necessary expression of the foundationalist attitude : the very study of logic generates a strange posture, which perhaps does not make sense at all.
To sum up, certain paradoxical formulas might have just as much meaning as the distance to α Centauri via the hyperspace. What we try to express positively through the main thesis of ludics, *locativity* : the existence of a « logical

space » with which one cannot tamper.

1.1.6 Logic vs. Physics

We just alluded to general relativity ; the comparison with Physics is always of interest. For instance, up to 1900, the dominant physical paradigm was determinism in the style of Laplace, « everything can be computed from the initial position », which contains in fact two layers, one being about abstract determinism, the other being about our faculty of prediction, semantics and syntax so to speak. The work of Poincaré —later, the theory of *chaos*— definitely ruined the mere idea of prediction, this must be compared to incompleteness. But the comparison turns short here, for we should imagine a physics surely chaotic, but still deterministic, to match the present state of logic : in this physics, future is written in a Big Book —but this book is out of reach. Fortunately quantum physics left nothing of this frightening fantasy !

1.1.7 Subjective aspects of realism

Of course one cannot destroy realism by an act of will, there is a natural tendency to reify... An example : some people introduced long ago the idea of potential (vs. actual), and soon afterwards the set of all potentialities... with the result that 99% of the original idea vanished : the potentialities were *reified*, i.e., actualised. The task in that case would be to define « potential » in such a way that the set of all potentialities would not make sense... moreover the definition should be quite natural, otherwise why shouldn't nature shun such a thing ?

This tendency to reify explains why the current interpretation of quantum mechanics is not that convincing. Imagine a sort of *quantum philosopher* arguing with —say— Descartes : the poor guy would not get a chance... but he would be right whereas Descartes and his realism are wrong. The problem is that the global structure of realism is much elaborated and deeply rooted in our minds —especially Western minds.

Coming back to logic, the stronghold of realism is of course natural numbers. It is clear that any foundational reflection should —sooner or later— cope with integers, which are so deeply rooted in our (wrong) intuitions and our (wrong, but efficient) formalisms, that one may question the possibility of achieving anything in that direction. No doubt that this is a problem, but one can imagine indirect approaches, typically through the theory of exponentials, see section 5 below.

1.1.8 Hands tied in the back

« *100g pasta, 1 litre water, 10g salt. Add salt to boiling water. Add pasta, stirring occasionally. Cook for 10 minutes. Drain pasta keeping part of water.* » This recipe for *Penne Rigate* is taken from a current brand, *Pasta Barilla.* The obvious question is the meaning of the recommendations (interdictions). First observe that if we follow them, everything works ; but is there a reason to follow them, what is the reference of these rules ? For instance it is legitimate to question the idea of salting the boiling water : could we salt before boiling ? « NO : the salt must be put after »... at least it is what some people will say —in relation to the authority of *Zia Ermenegilda*, a sort of meta-pasta that you cannot taste but to which you must conform, Tarskian *cuisine* so to speak. This attitude is common in foundations, there are too many artificial restrictions. By the way, this was one of the major objections of Hilbert to Brouwer : intuitionism is a sort of mathematics with « a hand tied in the back »[3]. Nowadays, most attempts at foundations keep a hand in the back. Certain principles —induction, comprehension... — are forbidden under a principle ending with « ism ». There is no doubt that this sort of *bondage* achieves something, but another « ism » may do as well, perhaps better. How can we decide ?

If we exclude divine revelations, the only possibility consists in making things interact with *alter egos.* In the case of *pasta*, one alters the recipe and see whether it tastes the same. Typically, put the salt before boiling, you will notice no difference ; push the cooking time to 15mn and you get glue.

To sum up, restrictions are not out of a Holy Book, but out of *use.* And use is *internal*, i.e., homogeneous to the object.

1.2 The present state of foundations

1.2.1 The copyright

Realism says that truth makes sense independently of the way we access it. Accept this and you are bound to compare the « strengths » of various formalisms, i.e., the sizes of the metaturtles. However, commonsense should tell us that nobody has ever seen the class of all integers, not to speak of this « Book of Truth and Falsehood » supposedly kept by Tarski. All these abstractions, truth, standard integers, are handled through proofs —hence the « reality » may involve a re-negotiation of the intertwining between proofs and « truth » —whatever the latter expression means, perhaps nothing at all.

But nothing of the like has been so far done. The compulsory access to « foundations » is through a bleak Trinity *Semantics/Syntax/Meta* : this approach

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[3] Hilbert himself was even more drastic as far as his fantasmatic « metamathematics » was concerned !

owns the *copyright*, if you don't accept this, you are not interested in foundations, period[4]. The problem with the copyright is that it dispenses one from addressing the original question : « no new disturbing phenomenon will be disclosed, we concentrate on internal problems ». In an area such as foundations, this attitude should be called *scholastics*, from the medieval philosophers —or rather their followers— who kept on transmitting an ossified approach to logic. Clearly, if this is the only approach to foundations, one should do something else... by the way this is what the main stream of mathematics realised long ago.

1.2.2 The input of computer science

Traditional foundations were built as a « logic from mathematics ». By the end of last century emerged a « logic from computer science » with its peculiarities : the concept of external —ethereal— Truth (the Father) was no longer pregnant ; on the other hand —since the computer is a badly syntactic, bureaucratic, artifact— the Son (Syntax) became omnipresent. Of course foundations of computer science difficultly departs from the Trinity —as in the sportive saying : don't tamper with a losing team !—, but something goes wrong. The mismatch is conspicuous everywhere, think for instance of « operational semantics » : the semantics of the language becomes the way you use it, i.e., is syntactical. For instance lambda-terms are sometimes described as syntax and the normalisation (rewriting) process as semantics : the idea is far from being stupid ; but try to present it in Tarskian dressing !
The same can be said of the notorious *Closed World Assumption* « something is false when not provable », which corresponds to a *procedural* view of logic : negation is applied to the cognitive process itself and not to some « abstract contents »[5].

1.2.3 The input of proof-theory

Traditional proof-theory was versatile enough to be transferred *mutatis mutandis* from the desertic steppes of consistency proofs to theoretical computer science. Lambda-calculi, denotational semantics, game-theoretic interpretations... prompted a new universe, reasonably free from foundational anguish, and mainly dedicated to (abstract) programming languages. It now seems that

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[4] This sort of situation is not exceptional, remember Communism with its copyright on « Progress ».

[5] This correct procedural intuition was translated into a commutation of negation with provability and stumbled on incompleteness, halting problems : « non-monotonicity », « circumscription »,... tried to accommodate some procedurality inside classical logic : to figure out the disaster, imagine a horse-cart powered by a rocket.

enough has been gathered to venture a first synthesis —this is *ludics*— and to transfuse fresh blood into the anaemic foundational body.

1.2.4 The locative thesis

When I say « fresh blood », this is not rhetoric, I mean completely new ideas ; really shocking ones, the kind that receives at best only polite reactions. This is the case for the *locative thesis*, which says that usual logic is wrong, because it is « spiritual », i.e., abstracts from the location. What is location ? Location in logic consists in these apparently irrelevant details known as names of variables, occurrences ; these details are evacuated in the first page of textbooks —they are also the favourite topic for the hangover lecture of Sunday morning. Location is also present in *realisability*, e.g., through the first bit left/right of a disjunctive realiser, see equation (26), p. 38. However, realisability leaks so badly —for instance $\neg A$ is realisable when A is not realisable —a problem of empty pivot, see 2.1.3— that one would hesitate to draw any positive consequence from it[6].

Locativity induces, among others :

- A clear approach to subtyping, inheritance, intersection types.
- The discovery of *locative* operations. Usual (spiritual) operations are obtained from the locative ones by use of *delocations*, i.e., isomorphic copies which prevent interferences. The typical example is that of a conjunction whose locative form is $A \cap B$ and whose spiritual form is $\varphi(A) \cap \psi(B)$; compare $A \cap A = A$ with $\varphi(A) \cap \psi(A) \simeq A \times A$, see 4.1.4 for a precise statement.
- The individuation of second-order quantification as an autonomous operation, not a poor relative of the first order case. In particular second-order logic validates classically wrong formulas.

All this is the evidence that locativity is a positive feature of logic, and not an impossible mess.

But we must now unwind the process leading to locativity ; in the beginning stands the uncommon idea of a procedural logic.

1.3 Procedural logic

1.3.1 The subformula property

The domain of validity of usual completeness corresponds to closed $\mathbf{\Pi}^1$ formulas (roughly : first order formulas universally quantified over their predicate

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[6] It is fair to say that ludics is mostly about finding the right space of realisers.

symbols ; this terminology is slightly misleading, since the $\mathbf{\Sigma}^0_1$ formulas of arithmetic are $\mathbf{\Pi}^1$). Completeness is the identity between truth and provability for $\mathbf{\Pi}^1$ formulas, whereas incompleteness is the failure of this property for the dual class $\mathbf{\Sigma}^1$, which contains the $\mathbf{\Pi}^0_1$ formulas. Now if we formulate logic in sequent calculus, we discover that the subformula property holds for *the same class* $\mathbf{\Pi}^1$, and fails outside. What does this mean ? If we consider cut-free proofs, then all possible proofs are already there, there is no way to produce new ones. In other terms, the calculus is complete —nothing is missing. Observe that this completeness does not refer to any sort of model, it is an internal property of syntax. Such a property cannot be an accident, it should be given its real place, the first :

The subformula property is the actual completeness

In order to make sense of this, syntax has to get its autonomy, to become the main object of study, with no relation to anything like a preexisting semantics... and, in the absence of any umbilical link to a semantics, it will eventually lose its character of syntax, it will become a plain mathematical object.

1.3.2 The disjunction property

The idea of a procedural logic must be ascribed to early intuitionism : an existence theorem should *construct* a witness. The most spectacular consequence is the *disjunction property* « A proof of $A \vee B$ induces[7] a proof of A or a proof of B ». Disjunction therefore commutes with provability, i.e., applies to the cognitive process : to prove $A \vee B$ is to prove A *or* to prove B. Intuitionistic logic is *procedural* in the sense that it refers to its own rules, in contrast to classical logic which is realistic, i.e., refers to its own *meta*.

Procedurality is considered suspicious, since it opens the door to subjectivism, but this is a superficial impression[8]. For instance realism interprets $\vee$ by meta-$\vee$; depending on the weather, meta-$\vee$ can be classical, intuitionistic, or enjoy intermediate properties like $\neg A \vee \neg\neg A$: there are full handbooks[9] dedicated to such logics, all of them sound and complete with respect to their own meta (the same in boldface). On the other hand, if we try to enrich classical logic with a connective enjoying the disjunction property, we badly fail : this connective turns out to be the same as classical disjunction.

There are two positive features of procedurality, first it is absolute (it refers

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[7] Only a moron would state $A \vee B$ if he has obtained A, hence the statement only applies to *cut-free* proofs. The word « proof » will therefore be short for « cut-free proof ».

[8] Some style procedurality as « intensional », others as rubbish : they basically agree !

[9] Not quite about alternative disjunctions : it is too difficult to tamper with this connective ; but this is not the case with modalities if you see what I mean.

to concrete operations on proofs), whereas realism is relative (it refers to our intuition of the universe, i.e., to what we already have in mind). Second, most procedural interpretations will be inconsistent : it was a true miracle that a connective enjoying the disjunction property could be found, and the price was a drastic modification of logic. Even more difficult was the discovery of an involutive procedural negation —this was the main achievement of linear logic. We eventually discover that procedurality is more demanding than realism.

1.3.3 A logic of rules

Procedurality is not intensional rubbish, it is a change of viewpoint, corresponding to the lineaments of a *logic of rules.* The idea is that disjunction can be applied to the proofs (prove A or prove B) : it is an operation on the representation, not on the « world » —if there is anything like the world.
In ludics, the disjunction property appears as the completeness of disjunction —an internal version of completeness, closely related to the subformula property, see 4.1.6.

1.4 An example : Barbara

1.4.1 Old scholastics

Barbara is the familiar syllogism :

$$\dfrac{\forall x(Ax \Rightarrow Bx) \quad \forall x(Bx \Rightarrow Cx)}{\forall x(Ax \Rightarrow Cx)}\,{\scriptstyle Barbara}$$

Scholastic philosophers were basically concerned with the explanation of syllogisms by mutual reduction ; in particular the acronyms *Disamis, Celarent,. . .* contain information as to these reductions. After centuries of repetitive work, this sort of activity became suspicious —and the expression « scholastics » derogatory.

1.4.2 New scholastics

By the beginning of last century, Łukasiewicz explained *Barbara* —and all the other figures— by the transivity of inclusion. The usual reaction to such an explanation is that of the layman in front of one of the compressed automobiles of the late César : rubbish[10]. With some education, you learn politeness and eventually find some (well-hidden) virtues in the product. But, education or

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[10] Myself I remember being taught about inclusion ; three increasing potatoes were drawn on the board, but a comment from the teacher, a sort of Barbara-without-the-name, was still needed.

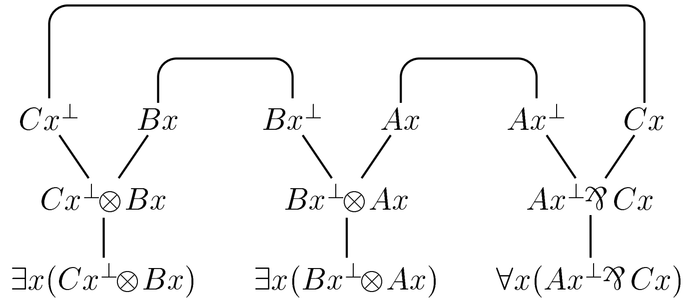
not, this regressive explanation fails at explaining the major point : why is there a rule, why this one precisely, how do we explain the distinctions between the various syllogisms —not to speak of their mutual reductions ?

1.4.3 Category theory

A major anti-realistic breakthrough was the introduction of *categories* in logic. The climateric work was the Curry-Howard isomorphism of 1969, but there was something in the air, think of Prawitz's work on natural deduction (1965), Scott domains (1969), my own system $\mathbb{F}$ (1970), Martin-Löf's type theory (1974). With the decisive input of computer science, the Boulder meeting (1987) was the apex of this « time of categories ».
Category-theory is often styled as nonsense because of the abuse of diagrams : indeed it is like Japanese *cuisine*, it does not stand mediocrity ; so let us forget the fat and proceed to the meat. Category-theory is a remarkable attempt at revoking this bleak distinction subject/object. In category, we have objects (objective) and morphisms (subjective)... and they live happily together.
Now look at the categorical version of the same : A, B, C become objects, proofs become morphisms, i.e., object and subject start to communicate. Eventually *Barbara* is composition of morphisms, something which is definitely not regressive with respect to Aristotle.

1.4.4 Barbara as a proof-net

Categories induced the first mature re-negotiation of the relation object/subject ; this eventually gave birth to *linear logic*. Linear logic yielded in turn a procedural explanation of *Barbara* rewritten as[11] :



The syllogism is eventually explained by the topology of this graph-like proof, called a *proof-net*.

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[11] Implication is translated as linear implication, not as usual implication... This is not an abuse, one had to wait for Boole to get the idea of idempotency, i.e., the contraction rule.

This interpretation is really subtler than the Tarskian one, for which all syllogisms are about transitivity of inclusion, period. In particular Tarskism cannot make sense of Aristotle's original taxonomy of syllogisms (first figure, second figure...). In contrast, proof-nets do : a recent analysis of syllogisms due to Abrusci shows that the figures of Aristotle correspond to crossing numbers : e.g., Aristotle « first figure syllogisms » induce planar proof-nets.

2 Hilbert revisited

The basic choice is between Hilbert's Programme (updated or not) —a sort of *millennium bug*— and anti-realism : something more experimental, more in the style of... 1900, when people were still trying, had not yet learned to pretend... Hilbert was precisely one of these guys, so let us revisit Hilbert.

2.1 On absurdity

2.1.1 Hilbert's program

Hilbert had to face certain paradoxes like Russell's. Since paradoxes come from our intuition of the real world, the idea was to put reality aside and to emphasise the formal treatment : this leads to another procedural approach to logic : *consistency.* We know since 1931 and the incompleteness theorem that this approach is wrong, but the refutation is rather technical. In what follows, we try to re-explain Hilbert's proposal as a duality between proofs of A and proofs of $\neg A$... with an essential flaw, namely that the pivot, the dualising object, (here : the proofs of absurdity) is empty, and it's quite impossible to build an interesting duality on an empty pivot, see below.

2.1.2 Duality in logic

Since *meaning by truth* is forbidden, let us try *meaning as use.* The use of a theorem is through its consequences : A is bound to have corollaries B, hence one should accept as true (i.e., as a theorem) anything having true consequences, whatever this means : A is true iff for any B, whenever $A \Rightarrow B$ is true, then B is true :

$$\frac{A \quad A \Rightarrow B}{B}\ \scriptstyle{\textit{Modus Ponens}}$$

This correct characterisation is not yet an explanation, think of $B = A$: one should eventually get rid of this unknown B, i.e., « close the system ». This prompts one to simplify the previous *Modus Ponens* into the cut :

$$\frac{\vdash A \quad A \vdash}{\vdash}\,{\scriptstyle Cut}$$

A proof of A is anything which, combined with a proof of $\neg A$ yields a proof of the absurdity. But what are the proofs of absurdity ?

2.1.3 Consistency

If we stick to the literal sense of « proof », there can be no proof of the absurdity. Then A is true (provable) if $\neg A$ is not true, i.e., not provable. In this monist reading, truth becomes the same as consistency. This is the basis of Hilbert's Programme, and the idea —the mistake— was rediscovered in computer science, through non-monotonic logics, closed world assumption... without even the excuse of novelty. Indeed, something in the structure of the duality should warn us in advance : the cut (*Modus Ponens*) establishes a duality between proofs of a formula and proofs of its negation, the output of the duality being in... the empty set. Transpose this in the Euclidian Space and think of the duality defined by $x \perp y := \langle x \mid y \rangle \in \emptyset$, i.e., x never orthogonal to y. Then $X^\perp$ is empty when X is non-empty, and $\emptyset^\perp = \mathbb{R}^3$, bleak indeed ! The empty pivot is responsible for this familiar drawback of realisability, namely that if A is not realisable, then $\neg A$ is realisable, thus destroying any reasonable hope of a faithful realisability interpretation... Not to speak of nonsenses like « $\neg A \vee \neg\neg A$ is realisable, but I cannot name the realiser », which are backstrokes of the Thief of Baghdad.

2.1.4 Wrong proofs

So far so bad... But isn't it because we used a simple-minded notion of proof too much linked with a literal interpretation of reasoning, in particular with the idea that formal reasoning should be correct ? Why not allow more « proofs » —maybe dubious— so as to replace the traditional duality :

$$\textit{Proofs} \text{ of } \vdash A \;/\; \textit{Models} \text{ of } A \vdash$$

with something of the form :

$$\textit{Proofs} \text{ of } \vdash A \;/\; \textit{Proofs} \text{ of } A \vdash$$

In this respect the proofs of the negation could be seen as sorts of « countermodels » —but now part of syntax, no longer of « reality ». The only *a priori* objection to such a thing is consistency ; but among proofs some might be more correct than others, so that consistency would be maintained by the exclusion of « wrong » proofs. By the way, if this idea of a wrong proof seems artificial to you, what would you say about transfinite stacks of metaturtles ? But where to find these additional wrong proofs ? Our only clue is the cut-elimination procedure : a proof of $\vdash A$ and a proof of $A \vdash$ put together should produce, through cut-elimination, a proof of $\vdash$[12].

2.1.5 Faith

Write naive set theory in natural deduction style, which Prawitz did in 1965. Russell's paradox yields a cut between a proof π of $\vdash a \in a$ and λ of $a \in a \vdash$ whose normalisation diverges, what we can note $\ll \pi \mid \lambda \gg = \Omega$; the symbol Ω —standing for divergence— is reminiscent of the u of recursion theory, or the $\Delta\Delta$ of lambda-calculus. Ω can be seen as a notational convenience, it can also be handled as an object, called *Faith* (i.e., the faith in convergence). But whatever choice we make, it is not in this way that we shall get a proof of the absurdity : our way out of Russell's paradox is not to forbid the cut between π and λ (which can be performed, this is precisely what Prawitz did), it is just to deny their *orthogonality* : if we consider syntax as a way to avoid divergence, π and λ cannot receive simultaneous specifications $\vdash A$ and $A \vdash$ (with the same A).

2.1.6 Daimon

We just saw that the basic duality can fail ; this means that it should sometimes succeed, but nothing in the extant proof-theoretic tradition can help us. Coming back to Euclidian space, $x \perp y := \langle x \mid y \rangle = 0$ is a beautiful orthogonality relation (the orthogonal of a line is a plane...) ; absurdity should be given a proof —something like 0—, but can we seriously allow that ?

What seemed absurd at the time of Hilbert is more reasonable at the time of proof-search : *Logic Programming* is organised as the search for cut-free proofs. Starting with the conclusion, the idea is to guess a last rule, then a rule above..., up to completion. Most likely the process eventually aborts, for want of a possible rule to apply, but we anyway did construct a truncated proof. It turns out that these truncated proofs are formal objects just as good as usual ones ; in particular one can develop a (straightforward) proof-theory and normalise cuts involving such proofs..., provided abortion is clearly acknowledged as a new rule, the *Daimon* ✠.

The addition of the daimon to syntax by no way produces an inconsistent system —provided we don't play on words. Of course every formula becomes

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[12] It is by the way funny to remark that traditional proof-theory is precisely about this situation —with the frustrating feature that the situation never occurs— : such a work about the empty set !

provable with the help of the daimon, but what about daimon-free provable formulas ? They are closed under consequence, i.e., cut-elimination : this is not surprising, before ludics, nobody ever heard about a logical daimon, but people knew how to normalise ; in other terms, the daimon cannot be created through normalisation.

Usual proofs therefore appear as daimon-free ones. The new « proofs » using the daimon occupy a space which is usually devoted to *models* : instead of having models of $\neg A$, we shall have proofs of $A^{\perp}$[13]. The difference is as follows :

- In the classical paradigm, the notions of proof and models are absolute. A proof of A proves A beyond discussion, a model of $\neg A$ refutes A beyond discussion. To the point that putting together a proof of A and a model of $\neg A$ is absurd. We are back to the empty pivot.

- In ludics, proofs —or models— are relative. They try their best to follow the rules, the truth tables, but nobody is perfect. An imperfect *proof* of A can be opposed to an imperfect *proof* of $A^{\perp}$; the interaction eventually yields a daimon —which has been produced by one of the two proofs, which is therefore incorrect. This does not necessarily make the other proof correct, since it may be opposed to another « counterproof » against which it now « fails ».

2.2 From proofs to designs

2.2.1 Geometry of proofs

What remains to be found is an object having the structure of a proof, but free from syntactical commitments. This object —a *design*— roughly corresponds to the geometrical structure underlying a sequent calculus proof. We should separate two layers, one being « what is actually performed », the other being useful comments ; useful to us —typically the name of the formula proven— but of no mathematical significance. A toy model of this is typed lambda-calculus : in a typed lambda-term the real guy is the pure lambda-term obtained by erasing the type decorations which can be seen as superfluous comments (they are useful specifications, but they don't participate to normalisation).

The process of finding the object has been very complex ; we already mentioned the invention of the Daimon. But the crucial breakthrough was the discovery of *polarities*.

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[13] Notice the use of linear negation ; this conceptual shift is technically impossible with usual negation.

2.2.2 Polarities

Curiously, the notion of polarity existed in logic long before its invention, but with no status. For instance, the « negative fragment » of intuitionistic logic regroups $\forall, \Rightarrow, \wedge$, namely the connectives which are well-behaved with respect to natural deduction. The notational gimmick of linear logic individualises two classes of connectives, one in « logical style », the negative $\parr, \&, \top, \bot, \ldots$ one in « algebraic style », the positive $\otimes, \oplus, 0, 1, \ldots$; in this way it is easy to memorise remarkable isomorphisms, typically distributivity. It is only around 1990 that this convenient distinction turned into a fundamental of logic.

The first remark is that, for any connective, there is always a side of the sequent on which contraction is free. Typically, the contraction rule on $\forall x A$ is redundant on the right of sequents : we can get it from a contraction on A ; the same holds for existence, but on the left. This is a first account of the distinction. The second remark comes from logic programming : negative connectives are *invertible*, i.e., they have a deterministic right rule : typically, proving $\vdash A \Rightarrow B$ is quite the same as proving $A \vdash B$. The fundamental remark of Andreoli —known as *focalisation*— is that the *positive* connectives —the non-invertible ones— enjoy a dual property. This property can be stated as follows : a cluster of operations of the same polarity can be seen as a single connective. Typically we can write complete rules for the double quantifiers $\forall x \forall y$ or $\exists x \exists y$ seen as a single operation, but not for the combinations $\forall x \exists y$ and $\exists x \forall y$. In the first case the two quantifications can be performed as a single step, in the second case, they must be performed sequentially. Polarity is therefore about time in logic, about the intrinsic causality between logical rules.

2.2.3 Polarised proofs

To understand how things work, let us take a concrete example, namely the positive formula $A = ((P^\perp \oplus Q^\perp) \otimes R^\perp)$, where P, Q, R are positive. A is treated as a single ternary connective $\Phi(P^\perp, Q^\perp, R^\perp)$ applying to *negative* subformulas $P^\perp, Q^\perp, R^\perp$, i.e, $A = \Phi(P^\perp, Q^\perp, R^\perp)$. We take advantage of the new notion of polarity and only use positive formulas ; this means that a negative formula is handled by means of its negation on the other side of the sequent. Our sequent calculus will therefore consists of sequents $\Gamma \vdash \Delta$ made of positive formulas. Inspection of the rules shows that it is enough to restrict to the case were Γ has at most one formula[14]. The rules for A are

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[14] This is quite the usual intuitionistic restriction, but left and right have been exchanged... This is because we are using *positive* formulas, whereas intuitionistic logic is merely concerned with negative formulas.

$$\frac{\vdash \Lambda, P, R \quad \vdash \Lambda, Q, R}{A \vdash \Lambda}\,(A\vdash,\{\{P,R\},\{Q,R\}\}) \qquad \begin{array}{c} \dfrac{P \vdash \Gamma \quad R \vdash \Delta}{\vdash \Gamma, \Delta, A}\,(\vdash A,\{P,R\}) \\[2ex] \dfrac{Q \vdash \Gamma \quad R \vdash \Delta}{\vdash \Gamma, \Delta, A}\,(\vdash A,\{Q,R\}) \end{array}$$

The right rules are obtained by combining a right Tensor-rule with one of the two possible right Plus-rules and negation, yielding two possibilities distinguished as $(\vdash A, \{P, R\})$ and $(\vdash A, \{Q, R\})$; *the* left rule is obtained by combining *the* Par-rule with *the* With-rule and negation. The rule is written $(A \vdash, \{\{P, R\}, \{Q, R\}\})$ in order to stress the existence of two premises, one involving P, R, the other involving Q, R.

As already explained, besides these « standard » rules, we need another one, the Daimon :

$$\frac{}{\vdash \Delta}\,\maltese$$

The daimon is restricted to the case « Γ empty » ; the case $A \vdash \Delta$ is indeed derivable, see the *Negative Daimon* in 3.2.2.

2.2.4 Normalisation

Proof-theory is organised along cut-elimination. In the previous case, let us assume that we are given a cut between $\vdash \Gamma, \Delta, A$ and $A \vdash \Lambda$:

1. If $\vdash \Gamma, \Delta, A$ has been obtained through a rule $(\vdash A, \{P, R\})$, replace with two cuts between $\vdash \Lambda, P, R$, $P \vdash \Gamma$ and $R \vdash \Delta$.
2. If $\vdash \Gamma, \Delta, A$ has been obtained through a rule $(\vdash A, \{Q, R\})$, replace with two cuts between $\vdash \Lambda, Q, R$, $Q \vdash \Gamma$ and $R \vdash \Delta$.
3. If $\vdash \Gamma, \Delta, A$ has been obtained by a daimon, replace with the proof of $\vdash \Gamma, \Delta, \Lambda$ consisting of a daimon.
4. Otherwise, apply a commutation of rules.

In the first two cases, the point is that the left rule for A is invertible, hence we do know that $A \vdash \Lambda$ follows from the two premises $\vdash \Lambda, P, R$ and $\vdash \Lambda, Q, R$.

3 Designs and behaviours

What follows is a slightly simplified version of ludics.

3.1 Designs

3.1.1 Locations

It remains to remove logical decorations. However, since a proof is basically a sequence of formulas, one should be careful not to remove everything. Indeed we shall keep the location, the *locus* of the formula. This locus is a very concrete notion, it is the place where the name of the formula is written : we shall assume that this space is the usual infinitely branching tree. To come back to our previous example, if A has been assigned the locus σ, then its immediate subformulas P, Q, R will be distinguished by *biases* $3, 4, 7$ —e.g., Q is the subformula of relative location 4 of A— and be respectively located in $\sigma * 3, \sigma * 4, \sigma * 7$. The logical rules just written will be interpreted by the disintegration of σ into its *subloci* $\sigma * 3, \sigma * 7$ and/or $\sigma * 4, \sigma * 7$.

3.1.2 Occurrences

It is time to revisit an old nonsense of logic, so-called « occurrences » : a given formula A may « occur » at different places. In theology this familiar property of Saints is known as *bilocation* but we are in mathematics : two objects with distinct locations cannot be quite the same. But they can be isomorphic ; concretely an isomorphism —called a « delocation »— relating the two « occurrences » is provided. This means that :

1. Our formula A can be located everywhere in our tree of loci. But when we change the location we are not quite with the same A, we are with an isomorphic copy.

2. The three subformulas of A have been given the relative locations $3, 4, 7$; we could have chosen as well $9, 6, 22$. The result would have been definitely different since for instance $7 \neq 22$, but of course isomorphic.

This distinction between equality and isomorphism is not a gilding of the lily. It corresponds to the replacement of the dominant *spiritual* treatment of logic with a more refined *locative* approach. It has spectacular positive consequences, such as the expression of the cartesian product as a delocated intersection, see 4.1.4.

3.1.3 Pitchforks

A sequent made of loci is called a *pitchfork*[15]. In a *negative* pitchfork $\xi \vdash \Upsilon$, ξ is called the *handle*, in a *positive* pitchfork $\vdash \Upsilon$, there is no handle, only *tines*.

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[15] These loci must be pairwise incomparable with respect to the sublocus relation.

What has been so far sketched translates into a sort of « pitchfork » calculus, with the following rules :

Positive rule : I is a ramification, i.e., a non-empty finite set of biases, for $i \in I$ the Λ_i are pairwise disjoint, with $\Lambda = \bigcup \Lambda_i$: one can apply the rule (finite, one premise for each $i \in I$)

$$\frac{\ldots \xi * i \vdash \Lambda_i \ldots}{\vdash \Lambda, \xi}\ {\scriptstyle (\vdash \xi, I)}$$

Negative rule : $\mathcal{N}$ is a set of ramifications, the *directory of the rule* : one can apply the rule (perhaps infinite, one premise for each $I \in \mathcal{N}$; as usual, $\xi * I$ is short for $\{\xi * i; i \in I\}$)

$$\frac{\ldots \vdash \Lambda, \xi * I \ldots}{\xi \vdash \Lambda}\ {\scriptstyle (\xi \vdash, \mathcal{N})}$$

Daimon :

$$\frac{}{\vdash \Lambda}\ \maltese$$

ξ is called the *focus* of the rules $(\vdash \xi, I)$ and $(\xi \vdash, \mathcal{N})$. The three rules discovered in 2.2.3, are therefore written $(\xi \vdash, \{\{3,7\},\{4,7\}\})$, $(\vdash \xi, \{3,7\})$ and $(\vdash \xi, \{4,7\})$.

3.1.4 Designs

A design is anything built in the pitchfork calculus by means of these three rules. No assumption of finiteness, well-foundedness, recursivity, is made ; designs can therefore be badly infinite. However infinite designs can naturally be written as « unions » of finite designs : any design $\mathfrak{D}$ is the directed « union » of the finite designs obtained by restricting all negative rules of $\mathfrak{D}$ to finite directories, all but a finite number of them being empty, see 3.2.4.
Usual sequent calculus contains a distinguished rule, called « identity axiom ». In usual syntax, identity axioms can be replaced with η-expansions, but they are still needed for propositional atoms (variables). In ludics, the η-expansion can proceed « beyond the atoms », yielding the *Fax*, see 4.2.7 ; this is why ludics has nothing like an identity axiom.

3.2 The analytical theorems

3.2.1 Normalisation revisited

In 2.2.4 we defined —or rather sketched— normalisation. This translates *mutatis mutandis* to the « pitchfork calculus », provided we avoid some pitfalls :

1. *Cut* is no longer a rule, it is a coincidence handle/tine between two designs. Typically a design of base (conclusion) $\sigma \vdash \tau$ and a design of base $\vdash \sigma$ that share σ form a *net* of base (conclusion) $\vdash \tau$[16].

2. Cases 1,2 of the syntactical process replace a cut on σ by two cuts on $\sigma * 3, \sigma * 7$ (or $\sigma * 4, \sigma * 7$). But this is a mere accident due to logic : we have respected the rules of the formula A and the left rule happens to exactly match the possible right rules. But there is no longer anything like A —for instance, σ does not encode the formula A—, which means that in case of a cut between a positive rule $(\vdash \sigma, I)$ and a negative rule $(\sigma \vdash, \mathcal{N})$ it may happen that $I \not\in \mathcal{N}$. In that case the normalisation process diverges, there is no normal form. We can use the symbol Ω to denote a diverging output, but this is a convenience, *Faith* not being a design.

3. We had in mind a finite normalisation, but designs need not be finite. In particular what may happen is an infinite sequence of normalisations, a cut is replaced with several cuts, one of these cuts in turn is replaced with other cuts... Such process also diverges, i.e., yields the non-design Ω as an output.

The value of the concept of design lies in a certain number of remarkable properties, which are the respective analogues of Böhm's theorem, Church-Rosser property, and stability.

3.2.2 Separation

Meaning is use ; if a design is a really meaningful structure, all of it must be usable, observable. But how do we use a design $\mathfrak{D}$ of base $\vdash \sigma$[17] ? Simply by cutting it with a *counterdesign* $\mathfrak{C}$ of base $\sigma \vdash$ and normalise. The base of the resulting net is the empty pitchfork $\vdash$ and there are very few possibilities :

Consensus : the normalisation converges, and the normal form is the only design of base $\vdash$, namely ✠. In that case, we say that $\mathfrak{D}$, $\mathfrak{C}$ are *orthogonal*, notation $\mathfrak{D} \perp \mathfrak{C}$.

Dissensus : the normalisation diverges, i.e., yields the non-design Ω.

Given $\mathfrak{D}$ one can consider the set $\mathfrak{D}^{\perp}$ of those counterdesigns $\mathfrak{C}$ which are consensual with $\mathfrak{D}$. The separation theorem states that designs are determined by their orthogonal, i.e., their use :

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[16] The notion is easily extended to several cuts : the coincidence graph must be connected/acyclic.

[17] The discussion applies to any base.

Theorem 1 (Separation)
If $\mathfrak{D} \neq \mathfrak{D}'$ then there exists a counterdesign $\mathfrak{C}$ which is orthogonal to one of $\mathfrak{D}, \mathfrak{D}'$ but not to the other.

This analogue of Böhm's theorem has a topological meaning. In the coarsest topology making normalisation continuous, the closure of the singleton $\mathfrak{D}$ is the biorthogonal $\mathfrak{D}^{\perp\perp}$. The separation theorem states that this topology is $\mathcal{T}_0$, i.e., that the preorder $\mathfrak{D} \preceq \mathfrak{D}' \Leftrightarrow \mathfrak{D}'^{\perp\perp} \subset \mathfrak{D}^{\perp\perp}$ is indeed an order. In other terms, we can form a Scott domain with designs : this is our bridge with first generation denotational semantics.
The separation theorem has an effective version, which amounts at giving an explicit description of the relation $\preceq$: $\mathfrak{D} \preceq \mathfrak{D}'$, i.e., « $\mathfrak{D}'$ is more converging than $\mathfrak{D}$ » if $\mathfrak{D}'$ has been obtained from $\mathfrak{D}$ by means of two operations :

Widen : add more premises to negative rules, i.e., replace $\mathcal{N}$ with $\mathcal{N}' \supset \mathcal{N}$.

Shorten : replace positive rules (ξ, I) with daimons —which has the effect of reducing the depth of branches.

On a positive base, the greatest —most converging element— is the Daimon ; Faith would be the smallest design —but has been excluded from the « official » definition. On a negative base, there is still a greatest design, called the *negative Daimon* :

$$\dfrac{\ldots \qquad \dfrac{}{\vdash \xi * I, \Lambda}\maltese \qquad \ldots}{\xi \vdash \Lambda}\,{\scriptstyle(\xi, \wp_f(\mathbb{N}) \setminus \{\emptyset\})}$$

The smallest design is called the *Skunk* :

$$\dfrac{}{\xi \vdash}\,{\scriptstyle(\xi, \emptyset)}$$

The name will be explained in 3.3.5.

3.2.3 Associativity

Strictly speaking, since normalisation is deterministic, there is no need for a Church-Rosser property. But besides the narrow technical meaning of Church-Rosser, there is a deeper one, namely that in the presence of two cuts, the output of normalisation is the same, whether we normalise them altogether, or one after the other, something like $ABC = (AB)C$:

Theorem 2 (Associativity)
Normalisation is associative : let $\{\mathfrak{R}_0, \ldots, \mathfrak{R}_n\}$ be a net of nets, then

$$[\![\mathfrak{R}_0 \cup \ldots \cup \mathfrak{R}_n]\!] = [\![[\![\mathfrak{R}_0]\!], \ldots, [\![\mathfrak{R}_n]\!]]\!] \tag{1}$$

Of course $[\![\mathfrak{N}]\!]$ is short for the normal form of the net $\mathfrak{N}$ and as in recursion theory, the equation also applies in case of divergence.
Associativity is often combined with separation to define adjoints, this is the *closure principle* « everything reduces to closed nets », where a closed net is a net whose base is the empty pitchfork. Typically, if $\mathfrak{D}, \mathfrak{E}$ are designs of respective bases $\xi \vdash \lambda$ and $\vdash \xi$, the normal form $[\![\mathfrak{D}, \mathfrak{E}]\!]$ is the unique design $\mathfrak{D}'$ of base $\vdash \lambda$ such that for every $\mathfrak{F}$ of base $\lambda \vdash$:

$$[\![\mathfrak{D}', \mathfrak{F}]\!] = [\![\mathfrak{D}, \mathfrak{E}, \mathfrak{F}]\!] \tag{2}$$

The normal form of a net $\mathfrak{S}$ is determined by the normal forms of all completions of $\mathfrak{S}$ into a closed net. The principle is very useful, since closed nets do not need commutative conversions.

3.2.4 Stability

Stability was introduced by Berry in the late seventies and is the distinctive feature of second generation denotational semantics —typically coherent spaces— which eventually led to linear negation.
Let us go back to separation and the relation $\mathfrak{D} \preceq \mathfrak{D}'$. The relation means that, if we replace $\mathfrak{D}$ with $\mathfrak{D}'$ in a converging net, then the resulting net still converges. As we saw it, there are two ways to be « more convergent », one is to be shorter, which can be summarised by $(\vdash \xi, I) \preceq \maltese$; the intuition is that if the normalisation process makes use of the positive action $(\vdash \xi, I)$, then the process will converge quicker if we replace it with $\maltese$: it immediately stops. The other way is to be wider, i.e., to replace a non-premise of a negative rule, which can be figured as a premise with nothing —i.e., the non-design Ω— above it, by the same premise with something —typically $(\vdash \xi, I)$— above it. The two ways are summarised by the equation :

$$\Omega \preceq (\vdash \xi, I) \preceq \maltese \tag{3}$$

Widening is of different nature since it does not alter converging normalisation processes (in contrast to shortening, which makes them… shorter) : Stability is about widening, which corresponds to a plain set-theoretic inclusion between designs[18]. Stability says that when $\mathfrak{D} \perp \mathfrak{E}$, then there are well-defined —i.e., minimum with respect to inclusion— finite $\mathfrak{D}' \subset \mathfrak{D}$, $\mathfrak{E}' \subset \mathfrak{E}$ which are responsible for this, i.e., such that $\mathfrak{D}' \perp \mathfrak{E}'$. This property is responsible for the major concept of ludics —incarnation, see 3.3.2.

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[18] Defined as *desseins*, i.e., sets of *chronicles* : this is indeed the official definition, the presentation as a pitchfork calculus being only a (slightly incorrect) convenience.

3.3 Behaviours

3.3.1 Formulas as specifications

A formula A can be —up to isomorphism— located everywhere, more precisely, when A is positive (resp. negative) it can receive any location $\vdash \sigma$ (resp. $\sigma \vdash$). Say for instance that A is negative and located in $\sigma \vdash$; then A will be identified with a certain set $\mathbf{G}$ of designs (representing the « proofs » of A) of base $\sigma \vdash$. This set is not arbitrary, it corresponds to a specification, a « how-to », i.e., a certain use of the designs. Remember that we reduced « use » to normalisation with counterdesigns : the use can therefore be represented by a set $\mathbf{G}^u$ of counterdesigns and we therefore obtain that $\mathbf{G} = \mathbf{G}^{u\perp}$. One can get rid of the arbitrary $\mathbf{G}^u$ by rewriting this as $\mathbf{G} = \mathbf{G}^{\perp\perp}$. The idea of formula as specification translates into :

Definition 1
A behaviour is a set $\mathbf{G}$ *of designs of a given base equal to its biorthogonal.*

Observe that $\mathbf{G}^{\perp}$, which is one of the possible choices for $\mathbf{G}^u$, is a behaviour too.
A behaviour is never empty : it contains the daimon of the right polarity — this is because $\maltese \perp \mathfrak{C}$ for all $\mathfrak{C}$. Behaviours are closed under $\preceq$: $\mathfrak{D}' \preceq \mathfrak{D}$ and $\mathfrak{D}' \in \mathbf{G}$, then $\mathfrak{D} \in \mathbf{G}$.

3.3.2 Incarnation

As a consequence, if $\mathfrak{D}' \subset \mathfrak{D}$ and $\mathfrak{D}' \in \mathbf{G}$, then $\mathfrak{D} \in \mathbf{G}$, but for « bad reasons » : nothing in $\mathfrak{D} - \mathfrak{D}'$ actually matters with respect to $\mathbf{G}$. Given $\mathfrak{D} \in \mathbf{G}$ there might be several $\mathfrak{D}' \subset \mathfrak{D}$ which are still in $\mathbf{G}$; but among those $\mathfrak{D}'$, there is a smallest one, the *incarnation* $|\mathfrak{D}_{\mathbf{G}}|$ of $\mathfrak{D}$ with respect to $\mathbf{G}$. The existence of the incarnation is just an alternative formulation of stability.
Incarnation is contravariant, i.e.,

$$\mathbf{G} \subset \mathbf{H} \Rightarrow |\mathfrak{D}|_{\mathbf{H}} \subset |\mathfrak{D}|_{\mathbf{G}} \tag{4}$$

Hence the incarnation of $\mathfrak{D}$ is maximum when $\mathbf{G}$ is the smallest (principal) behaviour $\mathfrak{D}^{\perp\perp}$ containing $\mathfrak{D}$; in this case $|\mathfrak{C}| = \mathfrak{C}$ (easy consequence of the separation theorem).

3.3.3 Formulas as games

Can we describe ludics as a « game semantics » ? Surely a design is a sort of strategy, which, when put against a counter-design, yields a *dispute* —the normalisation process, see 3.4.1— which is a play. Moreover, the use of the daimon in the dispute corresponds to giving up, hence we have a notion of winning. But these notions are defined once for all, there is no way to tamper with them. If we follow the usual game paradigm, little can be added : the first player can act in such a way that the opponent cannot win —what we called somewhere the *atomic weapon*. But wait a minute —the dispute generated by a design and a counterdesign needs not converge, i.e., yield a winner. We shall therefore impose the following : the two players do whatever they want, *provided they stay consensual*, i.e., that all disputes converge. This is the idea of a *game by consensus*.

The rule of the game $\mathbf{G}$ can then be specified by a set of counterdesigns $\mathbf{G}^r$, the only requirement being consensus, i.e., $\mathfrak{D}$ is an admissible strategy for $\mathbf{G}$ iff it is consensual with any $\mathfrak{E} \in \mathbf{G}^r$. If $\mathfrak{D} \perp \mathfrak{E}$ is short for consensus, this rewrites as $\mathbf{G} = \mathbf{G}^{r\perp}$, which implies $\mathbf{G} = \mathbf{G}^{\perp\perp}$; moreover there is a complete symmetry here, since $\mathbf{G}^{\perp}$ is the most natural $\mathbf{G}^r$. Eventually the game is explained by a consensus between players, as in real life —if real life is a game.

Of course a game by consensus can be seen as a usual game, with a rule, a referee, etc. In that respect, a design becomes a real strategy, but we have to work up to incarnation.

3.3.4 Typed vs. untyped

Designs can be handled in two ways :

Untyped : Designs can be considered as pure, i.e., *as themselves*.

Typed : Designs can be considered as part of a behaviour, i.e. with a restriction on their use ; they are no longer considered as themselves, but *as they should be*. In particular, they are only up to incarnation.

Typically, in the case of *subtyping*, i.e., of an inclusion $\mathbf{G} \subset \mathbf{H}$, the untyped viewpoint says that a design in $\mathbf{G}$ is a design in $\mathbf{H}$; the typed viewpoint induces a *coercion map* which replaces a design $\mathfrak{D}$ incarnated in $\mathbf{G}$, by its incarnation $|\mathfrak{D}|_{\mathbf{H}}$, see equation (4) above. This ambiguity is fundamental and has spectacular consequences —typically the *mystery of incarnation* which expresses the cartesian product as a delocated intersection, see 4.1.4.

3.3.5 The Skunk

Incarnation is delicate to grasp, so let us give a simple example. The set of all designs of a given base is a behaviour (the orthogonal of the empty set). When the base is negative, this behaviour is noted $\top$ and it corresponds to the additive neutral « true ». Now what is the *incarnation* of a design with respect to $\top$? This incarnation is the smallest design included in $\mathfrak{D}$ —and still in $\top$, but this additional requirement is always satisfied. This design is therefore the *empty design* $\mathfrak{Sk} = \emptyset$, corresponding to a first rule with an empty ramification :

$$\frac{}{\xi \vdash}\,{\scriptstyle(\xi,\emptyset)}$$

This design is called the *Skunk*, and the name is is a comment on his social life :

- $\top$ is the only behaviour containing $\mathfrak{Sk}$. From this viewpoint, the skunk is highly social, everybody lives in his company…
- … But they only pretend ; there is no *incarnated* design in $\top$ other than $\mathfrak{Sk}$.

Here we discover something essential, the biggest behaviour is also the smallest one, depending on the viewpoint —typed or untyped—we adopt. Incidentally, observe that $\top$ is the empty intersection and that $\{\emptyset\}$ is the empty product : this is indeed the 0-ary case of the *mystery of incarnation.*
By the way, the orthogonal of $\top$ is the smallest behaviour on a positive base. This behaviour is noted $\mathbf{0}$. As an easy consequence of separation, $\mathbf{0} = \{\maltese\}$.

3.4 Truth and completeness

3.4.1 Winning

If $\mathfrak{D} \perp \mathfrak{E}$, then the normalisation process —called a *dispute*— is a finite sequence $[\mathfrak{D} \rightleftharpoons \mathfrak{E}]$ consisting of the positive rules performed —up to a final daimon, which corresponds to termination. This is a sort of play, and the daimon corresponds to giving up ; but this daimon comes from one and exactly one of $\mathfrak{D}, \mathfrak{E}$ —this is stability— and this design *loses* the dispute. Let us say $\mathfrak{D}$ is *winning* when it never loses against any $\mathfrak{E}$; by separation, this is the same as saying that it does not use the daimon[19].

3.4.2 Truth

Winning induces a notion of *truth* : « a behaviour $\mathbf{G}$ is true when it contains a winning design, false when $\mathbf{G}^{\perp}$ is true ». Now, observe that, if $\mathfrak{D}, \mathfrak{E}$ are both winning, then they cannot be orthogonal —typically nobody gives up and the dispute becomes infinite. Hence it is impossible for a behaviour $\mathbf{G}$ to be both true and false. But it is not the case that a behaviour is either true or false.

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[19] The actual theory of winning involves two other conditions which are beyond the scope of this introduction ; but this one —called *obstination*— is by far the most important.

3.4.3 Completeness

Behaviour correspond to formulas, designs correspond to proofs. We have a rough idea of the translation of a proof into a design, and the next section on connectives will tell us about the translation of formulas into behaviours. Quite naturally, to each proof π of A we shall associate a design in the behaviour associated with A, $\boldsymbol{\pi} \in \boldsymbol{A}$. This is indeed a theorem, called *soundness*. The converse —i.e., whether or not this translation *leaks*— has been called *full completeness* by Abramsky : is every $\mathfrak{D} \in \boldsymbol{A}$, of the form $\mathfrak{D} = \boldsymbol{\pi}$ for some proof π of A ? The idea must be refined so as to avoid pitfalls :

1. Since full completeness admits the forgetful version « true implies provable », A must be restricted to those formulas for which usual completeness works. This class is familiar, it consists of first-order formulas —universally quantified over their predicate symbols to make them closed— the $\mathbf{\Pi}^1$ formulas.
2. Due to the daimon, $\mathfrak{D}$ must be winning.
3. Finally, $\mathfrak{D}$ must be incarnated... This last constraint is easy to understand : every design lives in the behaviour $\top$ which interprets the logical constant $\top$. But there is only one proof of $\top$, corresponding to the unique incarnated design of $\top$.

Full completeness is the statement :

THEOREM 3
If A is closed and $\mathbf{\Pi}^1$, if $\mathfrak{D} \in \boldsymbol{A}$ is a winning incarnated design, then $\mathfrak{D} = \boldsymbol{\pi}$ for some proof π of A.

The theorem has been established in [1] for second-order propositional logic without exponentials, i.e., the contraction-free part of logic.

3.4.4 Internal completeness

Full completeness is an important milestone, since it establishes the relevance of our approach —even if some connectives, basically exponentials, are still missing. But it is a step backwards too, since it reintroduces this duality syntax/semantics —this is the story of the Thief of Baghdad, see 1.1.4.

The tradition is to consider completeness as an external thing, syntax is (in)complete with respect to a given semantics. We saw in 1.3.1 that the subformula property should be considered as the real completeness « nothing is missing », and by the way what can be the value of a completeness which needs some external reference ? The task is to justify this mathematically, i.e., to formulate a sort of subformula property independently of any syntactical commitment.

Let us try at giving an internal meaning to full completeness. We start with a formula A. Logic yields rules which govern the syntactical proofs of A ; these proofs can be translated as a set $\mathbf{E}$ of designs ; what can we say about $\mathbf{E}$? Very little indeed, it is a set of designs of the same base ; to remember that it may come from the obedience to some rules, let us call such an arbitrary $\mathbf{E}$ an *ethics*. Now, we can see $\mathbf{E}^{\perp}$ as the (counter-) « semantics » of $\mathbf{E}$; now $\mathbf{E}^{\perp\perp}$ corresponds to what is validated by the « semantics » of $\mathbf{E}$. Completeness is therefore that $\mathbf{E} = \mathbf{E}^{\perp\perp}$, i.e., the fact that we have a direct description of a behaviour, without using the biorthogonal. For technical reasons —especially in the negative case—, the equality is required only up to incarnation : for instance the set $\{\mathfrak{Sk}\}$ is a complete ethics for $\top$.

The typical example of an internal completeness theorem is given by the connective $\oplus$, defined —modulo delocation— as $(\mathbf{G} \cup \mathbf{H})^{\perp\perp}$; one proves that the biorthogonal can be removed, i.e., that $\mathbf{G} \cup \mathbf{H}$ is a complete ethics for $\mathbf{G} \oplus \mathbf{H}$. By the way this internal completeness of $\oplus$ is the familiar disjunction property « a proof of $\mathbf{G} \oplus \mathbf{H}$ is a proof of $\mathbf{G}$ or a proof of $\mathbf{H}$ ».

Internal completeness is the essential ingredient of the proof of external (full) completeness. The task is, given an object of the appropriate type, to build a proof, indeed a cut-free proof. The main difficulty is of course to find the last rule, for we can then iterate the construction for the premises... Typically, in the case of a disjunction, the disjunction property provides one with the last rule, a left or right introduction of the disjunction.

4 The social life of behaviours

4.1 Additives

4.1.1 Locative vs. spiritual

All extant explanations of logic are *spiritual*. This means that the objects are taken up to isomorphism. This is plain in the Tarskian case —a formula refers to an interpretation somewhere on the Moon— this is also the case for more refined paradigms —typically category-theoretic ones. The spiritual treatment of logic can be smart enough to interpret conjunction as a cartesian product in the « constructive » case, and as a lunar intersection in general. But why this duality of interpretations ?... *Circulez, il n'y a rien à voir !* In fact the spiritual straightjacket makes it impossible to imagine a relation between an intersection and a cartesian product —by cartesian product, I mean a plain set-theoretic product, not a categorical nonsense. What is needed is the possibility of taking intersections of formulas —intersection types so to speak—, but how can we intersect sets defined up to isomorphism ?

Ludics is *locative* : a proof π of $\vdash A$, located in $\vdash \sigma$ interacts with a proof λ of $A \vdash$ located in $\sigma \vdash$, not on the Moon. A behaviour is made of precise objects, which are themselves, not an isomorphism class : in ludics we can take an « intersection of formulas », so let us see what happens. A *detour* through set theory is illuminating.

4.1.2 Locativity and set theory

The familiar operation of union admits two versions :

Locative : $X \cup Y$; this operation is commutative, associative, with neutral element $\emptyset$. But it is not spiritual, i.e., compatible with isomorphisms (here : bijections) ; in other terms $\sharp(X \cup Y)$ is not determined by $\sharp(X), \sharp(Y)$, the best we can say is $\sharp(X \cup Y) \leq \sharp(X) + \sharp(Y)$.

Spiritual : a.k.a. disjoint sum, $X + Y := f(X) \cup g(Y)$, where f, g are *ad hoc* injections. This is the total operation satisfying $\sharp(X + Y) = \sharp(X) + \sharp(Y)$. But commutativity, associativity, neutrality fail ; or rather you need a serious training in category-theory to understand in which way something of the like remains.

The same holds for the product, again with two possibilities :

Locative : $X \mathbin{\cup\mkern-12mu\times} Y = \{x \cup y; x \in X, y \in Y\}$; this operation is really commutative, associative, with neutral element $\{\emptyset\}$; it distributes over $\cup$. Unfortunately, we can only state that $\sharp(X \mathbin{\cup\mkern-12mu\times} Y) \leq \sharp(X).\sharp(Y)$.

Spiritual : a.k.a. product[20], $X \times Y := f(X) \mathbin{\cup\mkern-12mu\times} g(Y)$, where f, g are *ad hoc* injections. This is the total operation satisfying $\sharp(X \times Y) = \sharp(X).\sharp(Y)$. Again, commutativity, associativity, neutrality… only survive through (canonical) isomorphisms.

The locative product is —as far as I know— a novel operation ; it is quite good, think of $\wp_f(X \cup Y) = \wp_f(X) \mathbin{\cup\mkern-12mu\times} \wp_f(Y)$, an equality.

4.1.3 Delocation

In the case of set theory, f, g were chosen so as to make $f(X), g(Y)$, « disjoint » in an appropriate sense. Something similar can be done with ludics. The locus σ being fixed, delocations φ, ψ are defined by

$$\varphi(\sigma * i * \tau) = \sigma * 2i * \tau \qquad \psi(\sigma * i * \tau) = \sigma * 2i + 1 * \tau \tag{5}$$

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[20] An alternative definition of the set-theoretic product.

Since φ respects the tree structure, the image under φ of a design $\mathfrak{D}$ is easily defined and shown to be a design of the same base. We can also define the image under φ of a behaviour $\mathbf{G}$ as $\varphi(\mathbf{G}) = \varphi[\mathbf{G}]^{\perp\perp}$, with $\varphi[\mathbf{G}] = \{\varphi(\mathfrak{D})\ ;\mathfrak{D} \in \mathbf{G}\}$. In case the base is positive, this simplifies into $\varphi(\mathbf{G}) = \varphi[\mathbf{G}]$, but what about a negative base ? The answer is negative : if the first (downmost) rule of $\mathfrak{D}$ is $(\sigma \vdash, \mathcal{N})$, then the first rule of $\varphi(\mathfrak{D})$ is $(\sigma \vdash, \varphi(\mathcal{N}))$. But nobody forbids me to « widen » the ramification $\varphi(\mathfrak{D})$ into —say— $\varphi(\mathfrak{D}) \cup \{\{1\}\}$, so as to get $\mathfrak{E} \supset \varphi(\mathfrak{D})$, still in $\varphi(\mathbf{G})$, but clearly not in $\varphi[G]$. But wait a minute, $\mathfrak{E}\setminus\varphi(\mathfrak{D})$ is useless, i.e., does not contribute to the incarnation. In fact the incarnated designs of $\varphi(\mathbf{G})$ are all in $\varphi[\mathbf{G}]$, i.e., the equality $\varphi(\mathbf{G}) = \varphi[\mathbf{G}]$ holds up to incarnation : this is precisely what we called internal completeness.
In order to understand what φ, ψ actually achieve, observe that $\varphi(I) = \psi(J)$ iff $I = J = \emptyset$, which is impossible, since ramifications are non-empty. This makes the behaviours $\mathbf{G}' = \varphi(\mathbf{G}), \mathbf{H}' = \psi(\mathbf{H})$ *disjoint*, which means :

Positive case : the only design in $\mathbf{G}' \cap \mathbf{H}'$ is the daimon, i.e., $\mathbf{G}' \cap \mathbf{H}' = \mathbf{0}$.

Negative case : if $\mathfrak{D} \in \mathbf{G}' \cap \mathbf{H}'$, then $|\mathfrak{D}|_{\mathbf{G}'} \cap |\mathfrak{D}|_{\mathbf{H}'} = \emptyset$.

This is obvious, for instance, in the negative case, let $(\sigma \vdash, \mathcal{N})$ be the last rule of $\mathfrak{D}$, then $|\mathfrak{D}|_{\mathbf{G}'}$ is obtained by restricting $\mathcal{N}$ to $\{\varphi(I)\ ; \varphi(I) \in \mathcal{N}\}$, etc.

4.1.4 The mystery of incarnation

For any negative $\mathbf{G}', \mathbf{H}'$ of the same base $\sigma \vdash$:

$$|\mathfrak{D}|_{\mathbf{G}'\cap\mathbf{H}'} = |\mathfrak{D}|_{\mathbf{G}'} \cup |\mathfrak{D}|_{\mathbf{H}'} \tag{6}$$

which means that any incarnated design in $\mathbf{G}' \cap \mathbf{H}'$ is the union of an incarnated design in $\mathbf{G}'$ and an incarnated design in $\mathbf{H}'$. If we introduce the notation $|\mathbf{G}|$ for the set of incarnated designs of $\mathbf{G}$, this rewrites as :

$$|\mathbf{G}' \cap \mathbf{H}'| = |\mathbf{G}'| \mathbin{\cup\!\!\!\!\!\times} |\mathbf{H}'| \tag{7}$$

If $\mathbf{G}', \mathbf{H}'$ are disjoint, then the union (6) is disjoint ; then (7) rewrites as :

$$|\mathbf{G}' \cap \mathbf{H}'| \simeq |\mathbf{G}'| \times |\mathbf{H}'| \tag{8}$$

This might be the most spectacular result of ludics : the cartesian product is a particular case of the intersection, provided we focus on incarnation.

4.1.5 An example

Let us explain this in terms of the syntactical example of 2.2.3, i.e., consider the negative behaviour $\mathbf{K}$ corresponding to the sequent $A \vdash$. It will of course contain designs corresponding to $(A \vdash, \{\{P, R\}, \{Q, R\}\})$, but also designs corresponding to —say— $(A \vdash, \{\{P, R\}, \{Q, R\}, \{P, Q, T\}\})$, where T is another formula not related to A. But the incarnation of such a design will only retain the two premises $\{P, R\}, \{Q, R\}$. This shows something, namely that a negative rule can —in ludics— have useless premises. This is by the way the reason why full completeness is restricted to *incarnated* designs : the additional premises correspond to nothing syntactically visible.
Now $\mathbf{K}$ is the orthogonal of the set of proofs starting either with $(\vdash A, \{P, R\})$ or with $(\vdash A, \{Q, R\})$, i.e., it is the orthogonal of a union —which is the same as an intersection of orthogonals. We can therefore write $\mathbf{K} = \mathbf{G} \cap \mathbf{H}$. To take the incarnation in $\mathbf{G}$ (resp. $\mathbf{H}$) consists in retaining only the premise corresponding to $\{P, R\}$ (resp. $\{Q, R\}$). In that case the behaviours $\mathbf{G}, \mathbf{H}$ are disjoint, hence, up to incarnation, $\mathbf{K}$ appears as the cartesian product of $\mathbf{G}$ and $\mathbf{H}$.

4.1.6 The disjunction property

If $\mathbf{G}', \mathbf{H}'$ are positive behaviours of the same base $\vdash \sigma$, then the set (ethics) $\mathbf{G}' \cup \mathbf{H}'$ is not a behaviour. However, when $\mathbf{G}', \mathbf{H}'$ are disjoint

$$(\mathbf{G}' \cup \mathbf{H}')^{\perp\perp} = \mathbf{G}' \cup \mathbf{H}' \tag{9}$$

4.1.7 Additive connectives

Coming to the point of connectives —i.e., the social life of behaviours—, we discover that each connective can be presented —like the union and the product— in two different ways, a basic locative connective, and a spiritual one obtained by delocation from the basic case. Let us treat an example ; if $\mathbf{G}, \mathbf{H}$ are behaviours of base $\sigma \vdash$, then we can define the (negative) conjunction —i.e., a behaviour $\mathbf{K}$ of the same base—, in two different ways :

Locative : $\mathbf{K} = \mathbf{G} \cap \mathbf{H}$. This operation enjoys very good properties, typically (real) commutativity, associativity ; it has a real neutral element, namely $\top$. But completeness —internal, hence external— fails ; this is the well-known problem of the syntax of « intersection types ».

Spiritual : $\mathbf{G} \,\&\, \mathbf{H} = \varphi(\mathbf{G}) \cap \psi(\mathbf{H})$. The delocations φ, ψ are used to force the behaviours to be disjoint. As a result, we get a total and complete connective. But equalities are weakened into canonical isomorphisms.

The completeness of & is the fact that there is a simple description of $\mathbf{G} \mathbin{\&} \mathbf{H}$, *in terms of incarnation*, obtained from (8) :

$$|\mathbf{G} \mathbin{\&} \mathbf{H}| \simeq |\mathbf{G}| \times |\mathbf{H}| \tag{10}$$

Usual (negative) conjunction appears as a particular case of intersection ; the « intersection type » is more primitive.

If we turn our attention towards disjunction, then there are again two ways to form the disjunction of positive behaviours $\mathbf{G}, \mathbf{H}$, one being the locative sum $(\mathbf{G} \cup \mathbf{H})^{\perp\perp}$, the other being the spiritual sum $\mathbf{G} \oplus \mathbf{H} = (\varphi(\mathbf{G}) \cup \psi(\mathbf{H}))^{\perp\perp}$. In the spiritual case, the disjunction property enables us to remove the biorthogonal : this is the completeness of the sum, which can be written as :

$$\mathbf{G} \oplus \mathbf{H} \backslash \mathbf{0} \simeq (\mathbf{G} \backslash \mathbf{0}) + (\mathbf{H} \backslash \mathbf{0}) \tag{11}$$

4.1.8 Incarnation and records

This explanation of conjunction as a delocated intersection is indeed the final answer to a small mystery. In the eighties, computer scientists developed various theories of objects ; but they insisted on the point that records were not quite products. In a record, we have fields with *labels* ; in a cartesian product, we have two projections. In a record, we can decide to ignore part of the data, we still get a record of the same type : « a coloured point is still a point » ; the same is logically impossible, a pair (point,colour) is not a point. Ludics is very close to the « record spirit ». For instance, its locative features are the exact analogues of the field labels ; similarly, inheritance is a natural property of designs. It is projection, coercion (the fact for a point to lose its colour) that become more complex : coercion corresponds to incarnation « we have no use for the colour, let's erase it », and projection involves a delocation.

4.2 Multiplicatives

4.2.1 Shifts

As we said, one of the novelties of ludics is the use of polarity. This means that behaviours are divided into two classes, the positive and the negative ones. We just defined & as a connective sending negative behaviours to negative behaviours ; similarly $\oplus$ sends positive behaviours to positive behaviours. But the tradition is that connectives apply, independently of polarity. In order to allow this, it is enough to define connectives allowing a change of polarity : if $\mathbf{G}$ is of base $\sigma * 0$ (positive or negative), then $\updownarrow G$ is of base σ and of opposite polarity. More precisely, if $\mathbf{G}$ is of base $\sigma * 0 \vdash$, then $\downarrow\mathbf{G}$ is of base $\vdash \sigma$ (resp. if $\mathbf{G}$ is of base $\vdash \sigma * 0$, then $\uparrow\mathbf{G}$ is of base $\sigma \vdash$). The construction amounts

at adding to the bottom of each design in $\mathbf{G}$ a rule $(\vdash \sigma, \{0\})$ (resp. a rule $(\sigma \vdash, \{\{0\}\})$).
In terms of a naive game-theoretic intuition, the shift corresponds to a sort of dummy move. But this move is not quite dummy, because $\downarrow\uparrow \mathbf{G}$ is not isomorphic to $\mathbf{G}$: « operationally » speaking, two dummy moves have been added, with the possibility for each « player » to « give up » (i.e., use Daimon) at these early stages.

4.2.2 The locative tensors

The tensor is the positive product, the adjoint of (linear) implication. The general form of the definition is :

$$\mathbf{G} \circledast \mathbf{H} = \{\mathfrak{D} \circledast \mathfrak{D}' ; \mathfrak{D} \in \mathbf{G}, \mathfrak{D}' \in \mathbf{H}\}^{\perp\perp} \tag{12}$$

i.e., the tensor of behaviours is defined from a tensor of designs. The basic question is therefore : given any two designs $\mathfrak{D}, \mathfrak{D}'$ of base $\vdash \sigma$, how do we form their tensor product ? The answer depends on the respective first rules of $\mathfrak{D}, \mathfrak{D}'$:

1. If one of $\mathfrak{D}, \mathfrak{D}'$ is a daimon, then $\mathfrak{D} \circledast \mathfrak{D}' = \maltese$.

2. If the first rules of $\mathfrak{D}, \mathfrak{D}'$ are $(\vdash \sigma, I), (\vdash \sigma, I')$ and $I \cap I' = \emptyset$, then one can define $\mathfrak{D} \circledast \mathfrak{D}'$ as follows : the first rule is $(\vdash \sigma, I \cup I')$, with premises $\sigma * i$, for $i \in I$, on which we proceed like in $\mathfrak{D}$ and with premises $\sigma * i'$, for $i' \in I'$, on which we proceed like in $\mathfrak{D}'$.

3. But in case $I \cap I' \neq \emptyset$, there is no obvious answer. There is of course a spiritual answer : $\mathfrak{D} \otimes \mathfrak{D}' = \varphi(\mathfrak{D}) \circledast \psi(\mathfrak{D}')$. Since $\varphi(I) \cap \psi(I) = \emptyset$, we are back, up to a delocation, to the previous case.

But imagine that we want a *locative* tensor, something which should be to $\otimes$ what $\cap$ is to &. We have to solve a locative conflict —to take an exact image, imagine a flight, $\mathfrak{D}$ has booked rows $\{1, 12, 13\}$, $\mathfrak{D}'$ has booked rows $\{7, 13, 21\}$, with a conflict on the coveted row 13. The spiritual solution just mentioned amounts at delocating $\mathfrak{D}$ on rows $\{2, 24, 26\}$ and $\mathfrak{D}'$ on rows $\{15, 27, 43\}$, but this supposes an airplane big enough. If we turn our attention towards locative solutions, there are four possibilities —indeed four solutions admitting adjoints.

$\mathfrak{D} \olessthan \mathfrak{D}'$: $\mathfrak{D}$ gets seats $1, 12$, $\mathfrak{D}'$ gets $7, 13, 21$.

$\mathfrak{D} \ogreaterthan \mathfrak{D}'$: $\mathfrak{D}$ gets seats $1, 12, 13$, $\mathfrak{D}'$ gets $7, 21$.

$\mathfrak{D} \odot \mathfrak{D}'$: the flight is cancelled : $\mathfrak{D} \odot \mathfrak{D}' = \maltese$.

$\mathfrak{D} \obar \mathfrak{D}'$ **:** $\mathfrak{D}$ gets seats $1, 12$, $\mathfrak{D}'$ gets $7, 21$; 13 given to $\mathfrak{Sk}$. Better to travel with a skunk than not travelling at all !

The first two solutions are the symmetric expressions of the same non-commutative idea : one of the the two designs has priority, e.g., when $i \in I \cap I'$, proceed as in $\mathfrak{D}'$ (protocol $\olessthan$). The last protocol ($\obar$) gives the disputed row to anybody : when $i \in I \cap I'$, proceed as you want. For reasons of incarnation, « as you want » may be replaced with the « worst » case, the empty design $\mathfrak{Sk}$. But the most natural protocol remains $\odot$ which solves the conflict in a drastic way. These four protocols are associative, but this does not imply that the resulting tensor of behaviours is associative : if $\mathbf{G} \textcircled{c} \mathbf{H}$ is short for $\{\mathfrak{D} \circledast \mathfrak{D}'; \mathfrak{D} \in \mathbf{G}, \mathfrak{D}' \in \mathbf{H}\}$, there is no reason why $((\mathbf{G} \textcircled{c} \mathbf{H})^{\perp\perp} \textcircled{c} \mathbf{K})^{\perp\perp}$ should equal $((\mathbf{G} \textcircled{c} (\mathbf{H} \textcircled{c} \mathbf{K})^{\perp\perp})^{\perp\perp}$. Associativity comes from the existence of adjoints.

4.2.3 The adjoint implications

Each of the locative tensors has adjoints, for instance

$$\ll \mathfrak{F} \mid \mathfrak{D} \olessthan \mathfrak{D}' \gg = \ll \mathfrak{F}[\mathfrak{D}] \mid \mathfrak{D}' \gg = \ll [\mathfrak{D}']\mathfrak{F} \mid \mathfrak{D} \gg \tag{13}$$

$\mathfrak{F}, \mathfrak{F}[\mathfrak{D}], [\mathfrak{D}']\mathfrak{F}$ are of base $\sigma \vdash$ and $\mathfrak{D}, \mathfrak{D}'$ are of base $\vdash \sigma$. By the separation theorem, equation (13) *defines* —say— $\mathfrak{F}[\mathfrak{D}]$ in terms of $\mathfrak{F}, \mathfrak{D}$.
The adjoints enable one to give a characterisation of the orthogonals of the various tensors. Typically :

$$\mathfrak{F} \in (\mathbf{G} \olessthan \mathbf{H})^{\perp} \Leftrightarrow \forall \mathfrak{D}(\mathfrak{D} \in \mathbf{G} \Rightarrow \mathfrak{F}[\mathfrak{D}] \in \mathbf{H}^{\perp}) \tag{14}$$

This equation is used, together with the dual form :

$$\mathfrak{F} \in (\mathbf{G} \olessthan \mathbf{H})^{\perp} \Leftrightarrow \forall \mathfrak{D}'(\mathfrak{D}' \in \mathbf{H} \Rightarrow [\mathfrak{D}']\mathfrak{F} \in \mathbf{G}^{\perp}) \tag{15}$$

to prove the associativity of $\olessthan$ as well as its distributivity over the locative sum $(\mathbf{G} \cup \mathbf{H})^{\perp\perp}$. The connectives $\obar, \odot$ enjoy similar properties, and, of course, commutativity.

4.2.4 The spiritual tensor

$\mathbf{G} \otimes \mathbf{H}$ is defined as $\varphi(\mathbf{G}) \circledast \psi(\mathbf{H})$, where $\circledast$ is any of the locative tensors $\olessthan, \ogreaterthan, \obar, \odot$ (the choice is irrelevant). The spiritual tensor inherits all associativity, commutativity, distributivity from the locative case —but only up to isomorphism.

4.2.5 The meaning of commutativity

The fact that the spiritual tensor is not commutative is nothing but the fact that $f(x,y) \neq f(y,x)$. *A contrario*, the commutativity of the locative tensors $\obar, \odot$ mean something like $f(x,y) = f(y,x)$. Can we clarify this nonsense ? First, let us introduce the notation for the adjoint of $\odot$:

$$\ll \mathfrak{F} \mid \mathfrak{D} \odot \mathfrak{D}' \gg \; = \; \ll (\mathfrak{F})\mathfrak{D} \mid \mathfrak{D}' \gg \; = \; \ll (\mathfrak{F})\mathfrak{D}' \mid \mathfrak{D} \gg \qquad (16)$$

from which we actually get $((\mathfrak{F})\mathfrak{D})\mathfrak{D}' = ((\mathfrak{F})\mathfrak{D}')\mathfrak{D}$: the locative application is quite commutative ! In the case of usual (spiritual) application, we get $((\mathfrak{F})\varphi(\mathfrak{D}))\psi(\mathfrak{D}') \neq ((\mathfrak{F})\varphi(\mathfrak{D}'))\psi(\mathfrak{D})$, which explains the mystery. To sum up, locative application is commutative because the arguments are given *together with their locations* : there is no need for an order of application. You may think of the function as a module with several plugs : if application is plugging, then application is commutative. Delocation is indeed the possibility of shuffling plugs, no wonder that this induces a non-commutativity —but to some extent this non-commutativity is external to application.

4.2.6 Completeness properties

The completeness of the tensor consists in finding a complete ethics for $\mathbf{G} \circledast \mathbf{H}$; the obvious candidate is $\mathbf{G} \textcircled{c} \mathbf{H}$. To make the long story short, completeness holds in the case of $\varphi(\mathbf{G})\textcircled{c}\psi(\mathbf{H})$, which means that the spiritual conjunction $\mathbf{G} \otimes \mathbf{H}$ is complete. Contrarily to the additive case, this completeness result is highly non-trivial.

4.2.7 Sequents of behaviours

The behaviours so far considered have atomic bases $\vdash \sigma$ or $\sigma \vdash$. One can define behaviours on any base ; furthermore, one can form, on the model of sequents of formulas, sequents of behaviours. For instance, if $\mathbf{G}, \mathbf{H}$ are behaviours of respective bases $\vdash \sigma$ and $\vdash \tau$, one can define the behaviour $\mathbf{G} \vdash \mathbf{H}$ of base $\sigma \vdash \tau$ by :

$$\mathfrak{F} \in \mathbf{G} \vdash \mathbf{H} \Leftrightarrow \forall \mathfrak{D} \in \mathbf{G} \quad [\![\mathfrak{F}, \mathfrak{D}]\!] \in \mathbf{H} \qquad (17)$$

equivalently :

$$\mathfrak{F} \in \mathbf{G} \vdash \mathbf{H} \Leftrightarrow \forall \mathfrak{E} \in \mathbf{H}^{\perp} \quad [\![\mathfrak{F}, \mathfrak{E}]\!] \in \mathbf{G}^{\perp} \qquad (18)$$

The obvious question is « what about $\mathbf{G} \vdash \mathbf{G}$ and the *identity axiom* ? » First notice that the answer is —strictly speaking— negative : $\mathbf{G} \vdash \mathbf{G}$ has the base $\sigma \vdash \sigma$, which is not a pitchfork, remember that the loci must be pairwise incomparable. But we can reformulate the question with $\mathbf{G} \vdash \theta(\mathbf{G})$, where θ is the delocation $\theta(\sigma * \tau) = \sigma' * \tau$, and σ, σ' are incomparable.

Game-theoretically, the answer is well-known, it is the « copycat » strategy, which consists in recopying the last move of the opponent. Here we shall call it *Fax*, to stress the fact that it implements a delocation, here θ ; by the way, real copycats are at work everyday on the Web, and it is essential that they do their cheating at distance. But there is an even older intuition, namely the $\eta - expansion$ of the identity axiom, which reduces an identity on A to identities on P, Q, R, which in turn are reduced to identities… The process needs not stop : just understand that a propositional atom is indeed a variable, quantified universally or existentially ; once it has been replaced with a witness (an actual behaviour) the η-expansion can be resumed. The fax $\mathfrak{Fax}_{\sigma,\sigma'}$ is the following design :

$$\cfrac{\dots \qquad \cfrac{\begin{array}{c}\vdots\ \mathfrak{Fax}_{\sigma' * i, \sigma * i}\\ \dots\ \sigma' * i \vdash \sigma * i \dots\end{array}}{\vdash \sigma', \sigma * I}\;(\sigma', I)}{\sigma \vdash \sigma'}\;(\sigma, \wp_f(\mathbb{N}) \setminus \{\emptyset\})$$

The operationality of the fax is expressed by the following :

$$[\![\mathfrak{Fax}, \mathfrak{D}]\!] = \theta(\mathfrak{D}) \tag{19}$$

$$[\![\mathfrak{Fax}, \mathfrak{E}]\!] = \theta^{-1}(\mathfrak{E}) \tag{20}$$

$$[\![\mathfrak{Fax}, \mathfrak{D}, \mathfrak{E}]\!] = [\![\mathfrak{D}, \theta^{-1}(\mathfrak{E})]\!] = [\![\theta(\mathfrak{D}), \mathfrak{E}]\!] \tag{21}$$

The three equations (19) (20) (21) are equivalent definitions of the fax : this is plain from separation and associativity.

4.3 Quantifiers

4.3.1 Locative quantifiers

Locative quantifiers are quantifiers which do not follow the truth tables, which shun category theory. They are defined from the idea of intersection, not the idea of infinite product, and since no delocation can work, they are definitely different. Now the question is : does this become a mess, or do we get something nice out of it ? In fact something wonderful arises from the unexpected shock between locative quantification and spiritual connectives : these operations commute —sometimes beyond what seems reasonable, i.e., up to the violation of certain classical principles.

Let $\mathbf{G}_d, d \in \mathbb{D}$ be a family of behaviours of the same base ; then one defines the behaviours

$$\forall d \in \mathbb{D}\ \mathbf{G}_d := \bigcap_{d \in \mathbb{D}} \mathbf{G}_d \tag{22}$$

and

$$\exists d \in \mathbb{D}\ \mathbf{G}_d := (\bigcup_{d\in\mathbb{D}} \mathbf{G}_d)^{\perp\perp} \tag{23}$$

The typical example is second-order quantification : $\mathbb{D}$ consists in all the behaviours of a given positive base (typically $\vdash \langle\rangle$) ; if a formula $A[X]$ contains several « occurrences » of $X, X^\perp$, we interpret these occurrences as the images under delocations $\theta_1, \ldots, \theta_n$ of an unknown behaviour $\mathbf{G}$ (or its negation $\mathbf{G}^\perp$) based on $\vdash \langle\rangle$. The typical example $\forall X(X^\perp \parr \uparrow X)$, makes uses of the « occurrences » : $X^\perp = \theta(\mathbf{G}^\perp)$, with $\theta(\langle\rangle) = \langle\rangle$, $\theta(n * \sigma) = 2n * \sigma$, and $X = \rho(\mathbf{G})$, with $\rho(\sigma) = 1 * \sigma$.
Another example is the plain intersection type, $\mathbf{G}_0 \cap \mathbf{G}_1$; in that case, $\mathbb{D} = \{0, 1\}$. But first-order quantification is not locative, see below.

4.3.2 Prenex forms

The basic —and completely unexpected result— is that quantifiers commute with all *spiritual connectives* —but exponentials, not yet treated in ludics ; these commutations enable one to write *prenex forms*, a facility usually restricted to classical logic. Some of these commutations are obvious, they are just the result of polarity : $\forall$ commutes to negative, $\exists$ commute to positive, typically $(\forall X A X) \parr B$ is the same as $\forall X(AX \parr B)$. Other commutations are a real surprise ; let us give two examples :

$$\forall d(\mathbf{G}_d \otimes \mathbf{H}_d) = (\forall d\mathbf{G}_d) \otimes (\forall d\mathbf{H}_d) \tag{24}$$

This equation implies the second order formula $\exists X \forall Y(AX \Rightarrow AY)$.

$$\forall d(\mathbf{G}_d \oplus \mathbf{H}_d) = (\forall d\mathbf{G}_d) \oplus (\forall d\mathbf{H}_d) \tag{25}$$

This equation is even more violent, since it contradicts classical logic : typically it implies $\neg\forall X(X \vee \neg X)$. By the way this shows that a « constructive » —i.e., procedural— interpretation need not be weaker than a classical one : no hand tied in the back !

4.3.3 Locative phenomena

It is with second order quantification that locative phenomenons become the most prominent —shocking or promising, depending on one's attitude towards « foundations ».
Equation (25) can be understood from second-order realisability :

$$c \circledR A \vee B \;\Leftrightarrow\; \exists d\,((c = 1 * d \;\wedge\; d \circledR A) \vee (c = 2 * d \;\wedge\; d \circledR B)) \tag{26}$$

$$c \circledR \forall X A \;\Leftrightarrow\; \forall C\ c \circledR A[C/X] \tag{27}$$

From this :

$$c \circledR \forall X(A \vee B) \;\Leftrightarrow\; (c \circledR \forall XA) \vee (c \circledR \forall XB) \tag{28}$$

The reason is simple, if $c \circledR \forall X(A \vee B)$, then $c \circledR A[C_0/X] \vee B[C_0/X]$, hence is of the form $1 * d$, in which case $d \circledR A[C/X]$ for *all* C, or $2 * d$, in which case $d \circledR B[C/X]$ for *all* C. Observe that (27) refers to a quantification on something whose generic member has been called C, but we didn't need any information about those C but perhaps the fact that we can name one of them, C_0. In fact we are using the locative aspects of realisability : c is « located » among the $1 * d$ or among the $2 * d$, and this location is a feature of c not of the parameter C. The crucial point is therefore that the realisers c_C are *equal* (to c) ; should they be isomorphic, the property would fail. What we just explained is an exact rephrasing of the real argument in the archaic language of realisability : if we replace the maps $d \rightsquigarrow 1 * d$, $d \rightsquigarrow 2 * d$ with φ, ψ, c with a design, C with a positive behaviour, we get the correct proof of (25).

Of course, if somebody had stumbled 30 years ago[21] on something like (28) no conclusion would have been drawn —for realisability was leaking a lot, especially as soon as implication —or simply negation— was concerned : realisability was often percieved as « non-standard » ! But in ludics, the situation is quite different, there is no leakage up to $\mathbf{\Pi}^1$; a sequent like $\forall X(A \oplus B) \vdash (\forall XA) \oplus (\forall XB)$ is not $\mathbf{\Pi}^1$: it belongs to the incomplete realm where everything is possible, including a departure from classical logic. The usual rules for second order logic are correct, but incomplete ; the usual prejudice is to say « they are incomplete for want of enough comprehension axioms », and by the way it is true that, if we enlarge the syntax so as to get more comprehension, we get more $\mathbf{\Sigma}^1$ theorems. But here we got something which cannot be fixed in this way —remember that (25) implies the classically false $\neg\forall X(X \vee \neg X)$, that no comprehension axiom will entail.

The classical explanation of quantification is that of a big conjunction —maybe uniform in some sense. Classically speaking, $\forall X(A \oplus B)$ refers to each X separately. In ludics, this is not the case ; not because we absolutely want to produce warped effects, but because « there is not enough space ». If we want to interpret $\forall X(A \oplus B)$ as a sort of BIG conjunction, then we need disjoint delocations, one for each behaviour $\mathbf{G}$ for which the variable X stands. But there can be at most $\aleph_0$ such delocations[22], whereas the number of behaviours

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[21] Added in print : in fact it did happen, this is Troelstra's *uniformity principle* $\forall X\exists nA \Rightarrow \exists n\forall XA$, which relies —like (28)— on the fact that the realiser does not depend on the parameter C. This uniformity principle is genuinely locative and has been later studied by topos theoreticians, Lambek, P. Scott, Hyland, Rosolini... but rather as a warped by-product of « impredicativity » and not as a genuine principle of quantification omitted from predicate calculus.

[22] The same argument applies to old style realisability, there are more « types » than realisers : this is where « impredicativity » comes in, by forcing $\forall X$ to be an intersection ;

is $2^{2^{\aleph_0}}$.
The formula $\exists X \forall Y (AX \Rightarrow AY)$ is obtained from $\exists X \forall Y (AX \multimap AY)$ which in turn comes from $\forall X AX \multimap \forall Y AY$ by prenex operations, justified by (24). The typical element in this behaviour is a design obtained from the fax —that lambda-calculus would simply note $\lambda x.x$. Hence
$\lambda x.x \in \exists X \forall Y (AX \Rightarrow AY)$, but we cannot find any witness for X, i.e. the existence property fails. Concretely this means that we cannot remove the biorthogonal in the definition of the behaviour $\exists X \forall Y (AX \multimap AY)$, in other terms that this behaviour is incomplete. This is not a surprise, Gödel's theorem —or its version *ante litteram*, Cantor's theorem— forbids such a thing, but with a heavy argument based on diagonalisation : the paragon of incompleteness is Gödel's formula, which is $\mathbf{\Pi}^0_1$, i.e., $\mathbf{\Sigma}^1$. Nothing of the like here : there is no witness, period ; this is quite different from « I cannot name the witness ». So ludics does not enjoy the existence property... But is it a deadly sin ? Indeed the existence property is required for numerical quantifiers, i.e., quantifiers restricted to natural numbers —and of course it holds in that case.

4.3.4 First order quantification

First order quantification is a sort of big conjunction, maybe uniform in some sense : this is the viewpoint of model-theory, of German style proof-theory, etc. This is also the viewpoint we must adopt if we want to extend completeness to predicate calculus. It seems that the extant tools in ludics —esp. *uniformity*, a topic we avoided in this survey— are sharp enough to make it. But do we really need this, i.e., is there a need for a « constructive » —procedural— interpretation of predicate calculus ? To tell the truth, I never saw any such interpretation. In fact first order quantification has always been used in the particular case of *numerical* quantification, i.e., in contexts $\forall x\ (x \in \mathbb{N} \Rightarrow \cdot\,)$ and $\exists x\ (x \in \mathbb{N} \wedge \cdot\,)$. For instance coming back to realisability, one defines

$$c \circledR \forall nA \;\Leftrightarrow\; (\forall n\ \{c\}n \circledR A[n/x]) \wedge \ldots \text{[23]} \qquad (29)$$

But I never saw any *real* definition of $c \circledR \forall xA$.
It is therefore legitimate to question the interest of the predicate part of Heyting's logic, which has been unimaginatively modelled on classical predicate calculus. What would for instance be the result of treating first order quantification in the locative spirit, i.e., by means of equations (22) and (23) ? Obviously more formulas would be validated, with an incompleteness of the existential quantifier, typically $\exists x \forall y (Ax \Rightarrow Ay)$. But with no consequence of the form $\exists m \forall n (Am \Rightarrow An)$: this would not alter our beloved existence property for *numerical* quantification. $x \in \mathbb{N}$ translates into the usual Dedekind

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this in turn implies uniformity, which holds for any intersection.

[23] Some nonsense to fix the leakage.

formulation

$$\forall X(\forall z(z \in X \Rightarrow z+1 \in X) \Rightarrow (0 \in X \Rightarrow x \in X)) \tag{30}$$

The completeness of this Π^1 formula provides the required witness.

5 The challenge of exponentials

5.1 On integers

5.1.1 Kronecker

God created the integers, all else is the work of man ; the sentence is the best illustration of the intrinsic difficulty of foundations : how can we proceed beyond integers ? There is no way of making the natural number series lose its absoluteness : for instance model theory has considered *non-standard* integers, but the very choice of the expression betrays the existence of standard ones. Our approach to natural numbers must therefore be *oblique*, not because we want a warped notion, but because we meet a blind spot of our intuition. The possibility of an oblique approach is backed by quantum physics —by the way the major scientific achievement of last century— : one has been able to speak of non-realistic artifacts, in complete opposition to our « fundamental intuitions » concerning position, momentum, ... The short story of quantum mechanics begins with the discovery by Planck of small *cracks*, *asperities*, in this impressive realistic building, *thermodynamics*. Our only hope will be in the discovery of a « crack » in the definition of integers : even a small crack may do it.

5.1.2 Rates of growth

But it is quite desperate to seek an asperity in this desperately smooth $\{0, 1, 2, \ldots\}$, as long as we see it as a set. It becomes different if we see it as a « process », since we can play on the *rate of growth*. This is backed by the development of computational complexity, and paradoxically by the impossibility of proving any non-immediate separation theorem between the various classes : something about integers could be hidden there. The « rate of growth » could concern the possible functions involving integers ; more likely, there could be several rates of growth (polytime, logspace, ...) corresponding to different notions of integers.

5.1.3 Norms

Can we imagine different notions of integers ? Again, let us try an analogy : there is no problem with the finite dimensional space $\mathbb{C}^n$, corresponding to the basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$; but $\mathbb{C}^\omega$, which corresponds to the basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \ldots\}$ has no meaning at all —some quantitative information should be added, typically a norm, usually the ℓ^1-norm or the ℓ^2-norm. Could we find something like a « norm » (or norms) in the case of integers ?

5.1.4 Dedekind

There is no reason to criticise Dedekind's definition of integers (30), and little reasons to refuse its simplified version

$$\forall X((X \Rightarrow X) \Rightarrow (X \Rightarrow X)) \tag{31}$$

which is indeed the familiar polymorphic definition of the integers in system $\mathbb{F}$. We shall try to plug in real numbers in (31). Fortunately, $\forall X((X \Rightarrow X) \Rightarrow (X \Rightarrow X))$ is not flat like $\{0, 1, 2, \ldots\}$, it offers two asperities, one is $\forall X$: we could play on the possible X —i.e., restrict comprehension. This has been the main activity of an *ism* —predicative mathematics— : nothing ever came or can be expected from this approach, which is pure bondage. There remains the other entrance, the connective $\Rightarrow$.

5.2 On implication

5.2.1 The input of Scott domains

If we consider implication, there is a real crack in the building, not a very recent one, since it dates back to 1969, to Scott domains. This work was the final point to a problem illustrated by the names of Kleene, Kreisel, Gandy, ... on higher order computation[24]. The question was to find a natural topology on function spaces, in modern terms, to build a CCC of topological spaces. D. Scott (and independently Ershov) solved the question beyond doubt, but there is something puzzling : this is achieved by a restriction to queer spaces whose topology is not Hausdorff —only $\mathcal{T}_0$, the only separation property which costs nothing. In other terms, the Scott topology succeeds in keeping the cardinality of functionals quite low ; but it is cheap topology, in which separately continuous functions happen to be continuous.. This is our crack : the logical rules for implication contradict usual topology : we cannot interpret them with spaces like $\mathbb{R}$.
There are two possible interpretations of the absence of a convincing « continuous » explanation : the usual way is to fix it, and this is what everybody has so far tried. But there is another way out, maybe our logical rules, typically composition $f, g \rightsquigarrow f \circ g$ are *wrong*. Removing —more likely simply tampering

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[24] I have written somewhere that beyond second-order, higher order has been useful only for producing Ph.D. theses ; this was not quite true at that time.

with— composition may change a lot of things, and the kind of modification I am seeking is likely to have a different fate from intuitionism whose novelty was tamed by Gödel's translation : the « enemy » was still present —it only took the $\neg\neg$-disguise. We must accept the idea that perhaps the exponential $n \rightsquigarrow 2^n$ —which is a typical product of composition— may disappear once for all... This is shocking, but do we want something new, or are we happy with metaturtles ?

5.2.2 The input of linear logic

Linear logic started with a decomposition of implication :

$$A \Rightarrow B \;=\; !A \multimap B \tag{32}$$

into a more basic *linear implication* and a repetition operation, the *exponential* $!A$ which mainly takes care of contraction. The exponential-free part is something peculiar, extremely basic —so to speak foundationally neutral ; this is the fragment so far interpreted by ludics. Exponentials were excluded from the present version of ludics—not because of any essential impossibility— but because of a slight *malaise*, in particular as to their precise locativity : by the way, this malaise is part of the crack we are seeking.
Using linear logic, it was possible to revisit continuous interpretations :

- Formulas become Banach spaces[25], proofs linear maps of norm $\leqslant 1$.
- Positive connectives are interpreted by means of ℓ^1-norms ; typically the connective $\oplus$ corresponds to a direct sum equipped with $\|x \oplus y\| = \|x\| + \|y\|$.
- Negative connectives are interpreted by means of ℓ^∞-norms ; typically the connective & corresponds to a direct sum equipped with $\|x \oplus y\| = \sup(\|x\|, \|y\|)$.
- Usual implication $\mathbf{E} \Rightarrow \mathbf{F}$ corresponds to *analytical* functions from the open unit ball of $\mathbf{E}$ to the closed unit ball of $\mathbf{F}$.

The crack already noticed in Scott domains becomes more conspicuous : what prevents us from composing functions is the difference between an open ball and a closed ball —more precisely the impossibility of extending an analytic function to the closed ball in any reasonable way. For instance, for $\|x\| \leq 1$, the Dirac measures $!x$ do not depend continuously on x, $\|!x - !y\| = 2$ as soon as $x \neq y$.

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[25] Indeed *coherent* Banach spaces, in which the « dual space » is specified ; this accounts for the want of reflexivity of Banach spaces like ℓ^1, ℓ^∞.

5.3 Objective : exponentials

5.3.1 On infinity

Methodologically speaking, the introduction of exponentials induces a schizophrenia between labile operations —the basic linear connectives— and « stable » ones, the exponentials $!, ?$. One of the basic theses of linear logic was about infinity : infinity is not in some external reality, but in the possibility of *reuse*, i.e., contraction, i.e., exponentials :

Infinity as Eternity

For instance, no diagonalisation argument is possible in the absence of exponentials ; typically, Prawitz's naive set-theory, rewritten in a (basic) linear framework is consistent : this naive set theory normalises in logspace.

5.3.2 Light logics

In 1986 when linear logic was created, there was no question of departing from usual logic, to create yet another *Broccoli* logic. This is why the standard rules of exponentials have been chosen so as to respect intuitionistic logic through the translation (32). Infinity is concentrated in the exponentials, hence any tampering with exponentials will alter the properties of infinity. This is the oblique approach I mentioned : in this way, we can expect access to something beyond our realistic intuitions. Several systems with « light exponentials » have been produced ; my favourite being **LLL**, *light linear logic*, which has a polytime normalisation algorithm and can harbour all polytime functions.

Unfortunately these systems are good for nothing, they all come from bondage : artificial restrictions on the rules which achieve certain effects, but are not justified by use, not even by some natural « semantic » considerations.

5.3.3 Some science-fiction

Let us put things together. In the mismatch logic/topology, my thesis is that logic is wrong, not topology : so let us modify logic. Using linear logic, the modification must take into account the mismatch open ball/closed ball, and one can imagine several ways out —e.g., changing the diameters of balls— which all would have the effect of plugging real parameters in logical definitions such as Dedekind's.

However we are not yet in position to do so : the Banach space thing is only a *semantics*, i.e., a badly leaking interpretation. We have to import parameters —say complex— into ludics and accommodate exponentials in a continuous setting. We should eventually get a norm (rather several norms, none of them distinguished), not quite on the integers, but on a wider space... But the best programs are those written after their fulfilment.

References

[1] J.-Y. Girard. **Locus Solum**. *Mathematical Structures in Computer Science*, 11:301 – 506, 2001. [26]

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[26] This paper contains a very comprehensive bibliography, this is why we give no other reference here.