ON DENOTATIONAL COMPLETENESS

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Abstract

The founding idea of linear logic is the duality between A and $A^{\perp}$, with values in $\perp$. This idea is at work in the original denotational semantics of linear logic, coherent spaces, but also in the phase semantics of linear logic, where the « bilinear form » which induces the duality is nothing but the product in a monoid $\mathbb{M}$, $\perp$ being an arbitrary subset $\mathbb{B}$ of $\mathbb{M}$. The rather crude phase semantics has the advantage of being complete, and against all predictions, this kind of semantics had some applications. Coherent semantics is not complete for an obvious reason, namely that the coherent space $\mathbb{k}$ interpreting $\perp$ is too small (one point), hence the duality between A and $A^{\perp}$ expressed by the cut-rule cannot be informative enough. But $\mathbb{k}$ is indeed the simplest case of a Par-monoid, i.e. the dual of a comonoid, and it is tempting to replace $\mathbb{k}$ with any commutative Par-monoid $\mathbb{P}$. Now we can replace coherent spaces with « free $\mathbb{P}$-modules over $\mathbb{P}$ », linear maps with « $\mathbb{P}$-linear maps », with the essential result that all usual constructions remain unchanged : technically speaking cliques are replaced with $\mathbb{P}$-cliques and that's it. The essential intuition behind $\mathbb{P}$ is that it accounts for arbitrary contexts : instead of dealing with Γ, A, one deals with A, but a clique of Γ, A can be seen as a $\mathbb{P}$-clique in A. In particular all logical rules are now defined only on the main formulas of rules, as operations on $\mathbb{P}$-cliques. The duality between A and $A^{\perp}$ yields a $\mathbb{P}$-clique in $\mathbb{k}$, i.e. a clique in $\mathbb{P}$; strangely enough, one must keep the phase layer, i.e. a monoid $\mathbb{M}$ (useful in the degenerated case), and the result of the duality is a $\mathbb{MP}$-clique. We specify an arbitrary set $\mathbb{B}$ of such cliques as the interpretation of $\perp$. Soundness and completeness are then easily established for closed Π^1-formulas, i.e. second-order propositional formulas without existential quantifiers. We must however find the equivalent of $1 \in \mathcal{F}$ (which is the condition for being a « provable fact ») : a $\mathbb{MP}$-clique is *essential* when it does not make use of $\mathbb{M}$ and $\mathbb{P}$, i.e. when it is induced by a clique in $A^{\bullet}$. We can now state the theorem :

Let A be a closed Π^1 formula, and let a be a clique in the (usual) coherent interpretation $A^{\bullet}$ of A, which is the interpretation of a proof of A ; then a (as an essential clique), belongs to the « denotational fact » A° interpreting A for all $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$. Conversely any essential clique with this property comes from a proof of A.

1 Completeness

1.1 Classical completeness

The traditional semantical idea is to interpret formulas by some kind of models, thus yielding

- *Soundness* results : a provable formula is true in any model.
- *Completeness* results : a formula true in any model is provable.

This result holds of course for classical logic ; it can be extended to other logical systems : for instance intuitionistic logic is sound and complete w.r.t. Kripke models (and also w.r.t. topological models).

1.2 Phase semantics

A similar result holds for linear logic which is sound and complete w.r.t. phase semantics, see in particular [3], 2.1. Phase semantics is based on a commutative monoid $\mathbb{M}$, together with a distinguished subset $\mathbb{B}$ of $\mathbb{M}$; a formula will receive « truth values » in $\mathbb{M}$, i.e. a formula A will be interpreted as a subset A^* of $\mathbb{M}$. The crucial notion is the duality between A^* and $A^{\perp *}$: given $m \in A^*$, $m' \in A^{\perp *}$, then mm' should be a « truth value » for $\perp$, i.e. $mm' \in \mathbb{B} \quad (= \perp^*)$. One therefore defines *orthogonality* between objects of $\mathbb{M}$ by means of $m \perp m' \iff mm' \in \mathbb{B}$. In this respect the product behaves like a bilinear form ($\langle m, m' \rangle = mm'$) with value in $\mathbb{B}$, which plays the role of the « scalars ». The symmetry of linear logic (involutivity of negation) forces one to interpret any formula by a *fact*, i.e. by a subset X of $\mathbb{M}$ equal to its biorthogonal ; among all facts, the set $\mathbb{B}$ interprets the constant $\perp$ and the « bilinear » form can be seen as the canonical map from $X, X^\perp$ to $\perp$. The interpretation of connectives consists in building new facts from existing ones, and in particular the Tensor product $F \otimes G$ is defined as $(F.G)^{\perp\perp}$; the product is therefore used twice, both for the multiplicative conjunction and the duality.

Soundness, i.e. the fact that $1 \in A^*$ for all provable A, is proved without problems ; however this is a non-trivial result, since nothing in the notion of commutative monoid $\mathbb{M}$ and arbitrary subset $\mathbb{B}$ makes any reference to the peculiar laws of linear logic. *Completeness* is proved in a more *ad hoc* way : it consists in exhibiting, among all possible pairs, $\mathbb{M}$ the commutative monoid of contexts (i.e. multisets of formulas [1]) and among all possible $\mathbb{B}$ the set of all provable contexts. As usual completeness is slightly frustrating, the only positive point about it being that this particular choice of phase model is particularly inconspicuous in the theorem : « A is provable iff $1 \in A^*$ for any phase model $(M, \mathbb{B})$ » ; this stresses the fact that, if completeness is a desirable result, soundness should not be contrived.

By the way observe that completeness is by nature limited to a specific kind

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[1] On page 24 of [3] a footnote is missing after « (i.e. multisets of formulas », namely « We ignore the multiplicities of formulas $?A$, so that I is the set of contexts $?\Gamma$. »

of formulas : first-order formulas, and more generally second-order formulas in which the positive second-order universal (resp. existential) occur only positively (resp. negatively) : such formulas are called Π^1 (their negations are called Σ^1). Completeness fails for non-Π^1-formulas [2], hence its denotational extension, which implies usual completeness will be limited to Π^1-formulas.

1.3 Categorical completeness

However these extensions are not completely satisfactory, since we are dealing with constructive systems, for which the notion of proof is central : soundness and completeness only refer to the weaker notion of provability, i.e. w.r.t. models which can distinguish between two formulas, but not between two proofs of the same formula.

However there is a *semantics of proofs* whose general mathematical expression is categorical semantics : a proof of an implication $A \Rightarrow B$ is a morphism from the interpretation A^* of A to B^*. Categorical models of intuitionistic and linear logic associate different interpretations to distinct proofs of the same formula ; but to which extent are they complete ? In other terms, given a morphism from A^* to B^* is it the interpretation of a proof of the implication $A \Rightarrow B$? Up to now there is no satisfactory solution. Of course it is possible to give the abstract definition of an intuitionistic category (e.g. a CCC, i.e. a Closed Cartesian Category) and to prove some forms of completeness w.r.t. such categories, but it is easy to argue that a CCC is nothing but another presentation of intuitionistic logic, so what ? For the same reason one should reject, as contrived, any linear categorical completeness based upon « linear categories », i.e. upon the categorical axiomatization of linear logic.

1.4 Denotational completeness

We shall therefore limit ourselves to concrete categories, and we shall definitely work with *coherent spaces*, the original semantics of linear logic [3]. A general exposition of coherent semantics can be found in [3], see 2.2., from which we borrow the terminology and notations.

Starting with an assignment of coherent spaces to atomic formulas, one can associate a coherent space A^* to any formula A, and a clique $\pi^* \sqsubset A^*$ to any proof π of A. This is obviously the starting point for a soundness theorem, expressing that the rules of linear logic can be interpreted as operations on cliques of coherent spaces. But there is no obvious completeness counterpart, i.e. a result that would basically say that every clique in A^* is of the form π^* for some proof π of A :

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[2] This is one of the possible readings of Gödel's incompleteness, since the Gödel sentence G can be written $\forall x(\mathbb{N}(x) \Rightarrow F(x))$, with $F(x)$ a first-order and $\mathbb{N}(x)$ (which expresses that x is an integer) a second-order Π^1-formula : G, which is Σ^1 is true in any model (in fact : true) without being provable.

[3] Intuitionistic logic can be seen as a subsystem of linear logic, hence what we are doing applies also to intuitionistic logic.

- The empty set is always a clique in A^*, whereas the interpretation of a proof is usually nonempty.
- All constructions are usually infinite, but recursive in the parameters ; hence non-recursive cliques are not the interpretation of any proof.

In order to fix this failure, one must modify something in the interpretation, e.g. replace coherent spaces with something else, require some additional properties of the cliques etc. But this is a non-trivial endeavor ; in particular most modifications will accept the following extra principles :

- The *mix*-rule, namely the principle $A \otimes B \multimap A \parr B$.
- The identification between the two multiplicative neutrals 1 and $\perp$, both interpreted by a space $\mathbb{k}$ with one point.
- The identification between the two additive neutrals 0 and $\top$, both interpreted by an empty space.

1.5 The denotational duality

The only reasonable idea is to build a duality between X and $X^\perp$: there is a canonical bilinear map from $X \otimes X^\perp$ into $\mathbb{k}$, where $\mathbb{k}$ is the unit coherent space interpreting the constant $\perp$; concretely, if $a \sqsubset X$ and $b \sqsubset X^\perp$, then the clique $\langle a, b\rangle$ (which has at most one point) is the singleton $\mathbb{k}$ when $a \cap b \neq \emptyset$, and $\emptyset$ otherwise. The idea would be to select a set $\mathbb{B}$ of cliques in $\mathbb{k}$, and to define $a \perp b \iff \langle a, b\rangle \in \mathbb{B}$ when $a \sqsubset X$ and $b \sqsubset X^\perp$; a formula would therefore be interpreted by a *denotational fact*, i.e. a set of cliques in X equal to its biorthogonal.

The idea is not too bad, but it eventually fails for want of suitable $\mathbb{B}$ (only four possible choices). For instance, if $\mathbb{B}$ is empty, a denotational fact will either be empty or consist of all cliques in X ; on the other hand, if $\mathbb{B}$ is non-empty, we must accept the elements of $\mathbb{B}$ as the interpretation of proofs of $\perp$, and more generally that both A and $A^\perp$ might have proofs, which goes against completeness… unless we admit that cliques of A^* which are accepted will eventually be refused when completeness is at stake. So among the elements of a denotational fact it is necessary to distinguish between two classes of citizens, the higher kind, *essential* cliques being the subclass to which completeness applies… but there is no immediate way to make such a distinction.

1.6 Expanding the category

The solution comes from a close examination of the completeness argument w.r.t. phase semantics : one introduces the monoid of provable contexts. But since we are replacing « provability » with « proofs », one should instead consider the set $\mathbb{B}$ of proofs of arbitrary contexts Γ (or rather their denotational interpretation). Indeed one can build a gigantic coherent space $\mathbb{P}$, a kind of « infinite $\parr$ » of all coherent spaces A^*. As to its structure, $\mathbb{P}$ is a kind of monoid, exactly a

Par-monoid, i.e. it is equipped with a « Par-multiplication » : $\mathbb{P} \parr \mathbb{P} \multimap \mathbb{P}$ and a « Par-neutral » : $\mathbb{k} \multimap \mathbb{P}$ [4]. $\mathbb{B}$ can be seen as a set of cliques in $\mathbb{P}$. Now the basic idea is to replace plain proofs of A (seen as cliques in A^*) with proofs of Γ, A (seen as cliques in $\mathbb{P} \parr A^*$). The duality between X and $X^{*\perp}$ becomes a duality between $\mathbb{P} \parr X$ and $\mathbb{P} \parr X^\perp$: given a clique $a \sqsubset \mathbb{P} \parr X$, a clique $b \sqsubset \mathbb{P} \parr X^\perp$, the interpretation of the cut-rule yields a clique $c \sqsubset \mathbb{P} \parr \mathbb{P}$, which can be mapped by « Par-multiplication » to a clique $\langle a, b\rangle \sqsubset \mathbb{P}$. The basic orthogonality is therefore $\langle a, b\rangle \in \mathbb{B}$, and one can elaborate the semantics on this basis (i.e. a denotational fact is set of cliques in $\mathbb{P} \parr X$ equal to its biorthogonal, etc.). Since $\mathbb{B}$ is far from being empty, a denotational fact will hardly be empty, hence not all inhabitants should compete when completeness is at stake ; but if we restrict to inhabitants that are induced by a clique in X by means of the « Par-identity », then we obtain completeness ; indeed those cliques correspond to proofs with empty contexts.

But remember that completeness should not be achieved at the price of a contrived soundness ; fortunately, we can forget our particular $\mathbb{P}$ and $\mathbb{B}$ and observe that the interpretation works without any hypothesis on them, just as soundness w.r.t. phase semantics works for arbitrary $\mathbb{B}$. It remains to give a status to our use of a Par-monoid and the answer is extremely simple : all usual notions of linearity are replaced with $\mathbb{P}$-linearity, the familiar case being nothing more than the case $\mathbb{P} = \mathbb{k}$. This is clearly analogous to the replacement of commutative groups with R-modules, the ground case being the case $R = \mathbb{Z}$.

The fact that certain proofs have empty interpretations forces one to slightly complicate this very simple pattern : an additional commutative monoid $\mathbb{M}$ (which only matters in the case of empty cliques) must also be introduced. This copes with the degenerated cases of coherent semantics, i.e. empty cliques, in which the denotational information is absent, which forces one to deal with « truth values ». The modification induced by the auxiliary monoid to coherent semantics is modest, almost invisible, and our redaction tries to forget about it ; but it is a natural modification, involving a notion of $\mathbb{M}$-linearity with very satisfactory properties.

1.7 What has been achieved ?

This is always a delicate question, when we speak about completeness. For instance the first reaction of Yves Lafont in September 86 to phase semantics was something like « abstract nonsense », whereas later developments (including recent works by Lafont) suggest a less severe judgement. For the same reason, one should not be too harsh against the use of abstract monoids and abstract Par-monoids : eventually some application of this abstract nonsense will be found. Moreover, conceptually speaking, the individuation of a structure of « module » over a monoid and/or a Par-monoid induces an additional dimension in denotational semantics, which was obviously missing.

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[4] The typical Par-monoids are spaces $?X$.

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2 The category COH($\mathbb{MP}$)

Our goal is to introduce the category **COH**($\mathbb{MP}$) in two steps by adjoining two rather independent parameters, $\mathbb{M}$ and $\mathbb{P}$. In fact the usual category of coherent spaces will appear as the particular case **COH**($\mathbb{Ik}$), whereas the sole addition of $\mathbb{M}$ corresponds to **COH**($\mathbb{Mk}$), and the sole addition of $\mathbb{P}$ corresponds to **COH**($\mathbb{IP}$). **COH**($\mathbb{MP}$) is basically obtained by changing the notion of morphism, and this is why we shall concentrate on

- generalizing the notion of clique (from cliques to $\mathbb{MP}$-cliques) ;
- presenting morphisms from X to Y as functions sending $\mathbb{MP}$-cliques of X to $\mathbb{MP}$-cliques of Y and which are linear in a most straightforward sense ; by the way the $\mathbb{MP}$-cliques of X will be just the $\mathbb{MP}$-morphisms from $\mathbb{k}$ to X.

2.1 The category COH

We just content ourselves with an alternative description of linear maps :

Definition 1

A family (a_i) of cliques of X is said to be coherent *when its union is a clique ; in fact the family (a_i) is coherent exactly when the a_i are pairwise coherent. We use the notation $a = \sum a_i$ to mean that :*

- *a is the union of the a_i ;*
- *the (a_i) are coherent ;*
- *the (a_i) are pairwise disjoint.*

and we say that a is the coherent sum *of the a_i.*

Proposition 1

Let f be a function mapping cliques of X to cliques of Y ; then f is linear iff it preserves all coherent sums, i.e. iff $f(\sum a_i) = \sum f(a_i)$ for any coherent disjoint family.

Proof. — Assume that f preserves all coherent sums ; then

- $f(\emptyset) = f(\emptyset + \emptyset) = f(\emptyset) + f(\emptyset)$ forces $f(\emptyset)$ to be empty [5].
- Assume that a and b are compatible cliques ; then $a \cup b = (a-b)+(b-a)+a\cap b$, a coherent sum. $f(a) = f(a-b) + f(a \cap b)$, $f(b) = f(b-a) + f(a \cap b)$, $f(a \cup b) = f(a-b) + f(a \cap b) + f(b-a)$, hence $f(a \cup b) = f(a) \cup f(b)$.

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[5] This proof does not make use of sums of empty families.

- It is easily shown that f preserves all unions (not only finite ones).
- If a and b are compatible, then $f(a-b)$, $f(b-a)$ and $f(a\cap b)$ are disjoint, so $f(a\cap b)=(f(a-b)+f(a\cap b))\cap(f(b-a)+f(a\cap b))=f(a)\cap f(b)$.

Conversely stability implies that f preserves disjointness, and from preservation of coherent unions we immediately get preservation of coherent sums. □

2.2 The category COH($\mathbb{M}\Bbbk$)

In the sequel $\mathbb{M}$ is a commutative monoid, noted multiplicatively ; all definitions are trivial if $\mathbb{M}=\mathbb{I}$, the trivial monoid, i.e. **COH**($\mathbb{I}\Bbbk$) is **COH**.

Definition 2

A $\mathbb{M}$-clique in a coherent space X is a pair (m,a) of a point $m\in\mathbb{M}$ and a clique $a\sqsubset X$. Two $\mathbb{M}$-cliques (m,a) and $(m'a')$ are coherent *when $m=m'$ and $a\cup b$ is a clique. One defines in a similar way the union, the intersection, disjointness of two coherent $\mathbb{M}$-cliques ; one can also define the sum of a disjoint coherent family, provided the family is non-empty.*
The scalar multiplication of a $\mathbb{M}$-clique (n,a) by a scalar $m\in\mathbb{M}$ is defined as $m.(n,a)=(m.n,a)$.
A map f from $\mathbb{M}$-cliques of the coherent space X to $\mathbb{M}$-cliques of the coherent space Y is $\mathbb{M}$-linear when :

- *it preserves all coherent sums of non-empty families ;*
- *it preserves scalar multiplication, i.e. if $m\in\mathbb{M}$ and a is a $\mathbb{M}$-clique, $f(m.a)=m.f(a)$.*

Proposition 2

A $\mathbb{M}$-linear map from X to Y is of the form $f(m,a)=(m.m_0,g(a))$ with $m_0\in\mathbb{M}$ and g a linear map from X to Y. Conversely, given m_0 and g as above, the formula $f(m,a)=(m.m_0,g(a))$ defines a $\mathbb{M}$-linear map from X to Y.

PROOF. — $f(1,a)=(m(a),g(a))$ for a certain function g from X to Y which is easily shown to be linear. Moreover if $m(\emptyset)=m_0$, then it is immediate that $m(a)=m_0$; by linearity over $\mathbb{M}$ we get $f(m,a)=f(m(1,a))=(m.m_0,g(a))$. Conversely any function f of the given form is $\mathbb{M}$-linear. □
Hence $\mathbb{M}$-linear maps from X to Y are in bijection with $\mathbb{M}$-cliques in $X\multimap Y$, and composition of (m,f) with (n,g) is $(m.n,fg)$. More generally everything already done with coherent spaces adapts, *mutatis mutandis* to the new category. For instance the set of all $\mathbb{M}$-linear maps from X to $\Bbbk$ is isomorphic with the set of all $\mathbb{M}$-cliques in $X^\perp$.
The monoid essentially distinguishes between several (pairwise incoherent) empty sets, and a $\mathbb{M}$-clique is a usual clique plus an emptyset contained in the clique. We can see the monoid as a way to desingularize the additive neutrals (more

generally all the cases involving empty coherent spaces) : the duality between X and $X^{\perp}$, when X has an empty web takes all empty $\mathbb{M}$-cliques of $\mathbb{k}$ as values. In particular we see that the case where X has an empty web is already as rich as usual phase semantics.

2.3 Par-monoids

From now on we shall work as if the connective $\parr$ were literally associative, and similarly as if the neutral element $\mathbb{k}$ were literally the neutral element [6] ; this saves a lot of useless diagrams, and can be made rigorous anyway.

Definition 3

A (commutative) Par-monoid *is a coherent space $\mathbb{P}$ together with a linear map $\mathbf{c}$ from $\mathbb{P} \parr \mathbb{P}$ to $\mathbb{P}$, and a linear map $\mathbf{w}$ from the unit coherent space $\mathbb{k}$ interpreting $\perp$ into $\mathbb{P}$ (i.e. a clique in $\mathbb{P}$), enjoying the analogue of commutativity, associativity and neutrality, namely the commutativity of the diagrams :*

$$\begin{tikzcd}
\mathbb{P} \parr \mathbb{P} \arrow[rr, "\mathbf{c}"] \arrow[dr, "\sigma"'] & & \mathbb{P} \\
& \mathbb{P} \parr \mathbb{P} \arrow[ur, "\mathbf{c}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P} \parr \mathbb{P} \parr \mathbb{P} \arrow[r, "\mathbf{c} \parr \mathbb{P}"] \arrow[d, "\mathbb{P} \parr \mathbf{c}"'] & \mathbb{P} \parr \mathbb{P} \arrow[d, "\mathbf{c}"] \\
\mathbb{P} \parr \mathbb{P} \arrow[r, "\mathbf{c}"'] & \mathbb{P}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, "\mathbb{P}"] \arrow[dr, "\mathbb{P} \parr \mathbf{w}"'] & & \mathbb{P} \\
& \mathbb{P} \parr \mathbb{P} \arrow[ur, "\mathbf{c}"'] &
\end{tikzcd}$$

where σ stands for the « flip » between two copies of $\mathbb{P}$ and $\mathbb{P}$ stands for

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[6] Of course it is impossible to do the same with commutativity.

the identity map of $\mathbb{P}$. The last two diagrams make use of our conventions concerning associativity and neutrality. The map $\mathbf{c}$ is called Par-multiplication, or better contraction, *whereas the map $\mathbf{w}$ is called Par-neutral, or better* weakening.

Remark. — The typical example of a Par-monoid is a space $?X$ equipped with the maps corresponding to contraction and weakening. Among Par-monoids there is a distinguished one, namely the space $\mathbb{k}$ (with trivial maps $\mathbf{c}, \mathbf{w}$). Par-monoids have been introduced in [2] under the name « negative correlation spaces ». We prefer to now use a more transparent terminology : in this way a usual monoid would become a Tensor-monoid, and the dual of a Par-monoid, i.e. what is usually called a comonoid would become a Tensor-comonoid (what is usually called comonoid) whereas the dual of monoid would become a Par-comonoid. There is only one defect with this new terminology, namely that it does not stress enough the analogy with rings : the operation $\mathbf{c}$ behaves like a multiplication (and $\mathbf{w}$ like the unit) of a ring, the addition being mimicked by the sum of coherent cliques.

2.4 The category $\mathbf{COH}(\mathbb{I}\mathbb{P})$

In the sequel, $\mathbb{P}$ stands for a Par-monoid. All definitions are trivial when $\mathbb{P} = \mathbb{k}$, i.e. $\mathbf{COH}(\mathbb{I}\mathbb{k})$ is $\mathbf{COH}$, which is consistent with our previous notations.

Definition 4

Let X be a coherent space ; then a $\mathbb{P}$-clique in X is a clique $a \sqsubset X \parr \mathbb{P}$. In particular (remember our convention saying that $\mathbb{k}$ is literally neutral), a $\mathbb{P}$-clique in $\mathbb{k}$ is a clique $a \sqsubset \mathbb{P}$.

Definition 5

Let X and Y be coherent spaces ; a $\mathbb{P}$-morphism from X to Y is a function f mapping $\mathbb{P}$-cliques of X to $\mathbb{P}$-cliques of Y and enjoying the following :

- *f preserves arbitrary coherent sums ;*
- *consider the linear map $f \parr \mathbb{P}$ from $X \parr \mathbb{P} \parr \mathbb{P}$ to $Y \parr \mathbb{P} \parr \mathbb{P}$; it can be seen as a map from $\mathbb{P}$-cliques of $X \parr \mathbb{P}$ to $\mathbb{P}$-cliques of $Y \parr \mathbb{P}$: we require that $(Y \parr \mathbf{c})((f \parr \mathbb{P})(a)) = f((X \parr \mathbf{c})(a))$ for any $\mathbb{P}$-clique a of $X \parr \mathbb{P}$. In other terms the following diagram is commutative :*

$$\begin{tikzcd}
X \parr \mathbb{P} \parr \mathbb{P} \arrow[r, "f \parr \mathbb{P}"] \arrow[d, "X \parr \mathbf{c}"'] & Y \parr \mathbb{P} \parr \mathbb{P} \arrow[d, "Y \parr \mathbf{c}"] \\
X \parr \mathbb{P} \arrow[r, "f"'] & Y \parr \mathbb{P}
\end{tikzcd}$$

Proposition 3

$\mathbb{P}$-morphisms from X to Y are in natural correspondence with $\mathbb{P}$-cliques in $X \multimap Y$.

PROOF. — f is indeed a linear map from $X \parr \mathbb{P}$ to $Y \parr \mathbb{P}$. Now observe that $X \parr \mathbf{w}$ induces a canonical map from X to $X \parr \mathbb{P}$, so $f(X \parr \mathbf{w})$ is a linear map g from X to $Y \parr \mathbb{P}$. Using the isomorphism

$$X \multimap (Y \parr \mathbb{P}) \simeq (X \multimap Y) \parr \mathbb{P}$$

we can associate a $\mathbb{P}$-clique of $X \multimap Y$ with f.
Conversely, a $\mathbb{P}$-clique in $X \multimap Y$ induces a linear map g from X to $Y \parr \mathbb{P}$, hence a linear map $g \parr \mathbb{P}$ from $X \parr \mathbb{P}$ to $Y \parr \mathbb{P} \parr \mathbb{P}$, and using contraction, a map $(Y \parr \mathbf{c})(g \parr \mathbb{P})$ from $X \parr \mathbb{P}$ to $Y \parr \mathbb{P}$. This map f, seen as a map from $\mathbb{P}$-cliques to $\mathbb{P}$-cliques can be shown to be $\mathbb{P}$-linear, by means of the commutative diagram :

$$\begin{tikzcd}
X \parr \mathbb{P} \parr \mathbb{P} \arrow[r, "g \parr \mathbb{P} \parr \mathbb{P}"] \arrow[d, "X \parr \mathbf{c}"'] & Y \parr \mathbb{P} \parr \mathbb{P} \parr \mathbb{P} \arrow[d, "Y \parr \mathbb{P} \parr \mathbf{c}"] \arrow[dr, "Y \parr \mathbf{c} \parr \mathbb{P}"] & \\
X \parr \mathbb{P} \arrow[r, "g \parr \mathbb{P}"'] \arrow[dr, "f"'] & Y \parr \mathbb{P} \parr \mathbb{P} \arrow[d, "Y \parr \mathbf{c}"] & Y \parr \mathbb{P} \parr \mathbb{P} \arrow[dl, "Y \parr \mathbf{c}"] \\
& Y \parr \mathbb{P} &
\end{tikzcd}$$

The two operations are then easily shown to be reciprocal, by means of the commutative diagrams :

$$\begin{tikzcd}
X \arrow[rr, "X \parr \mathbf{w}"] \arrow[d, "g"'] & & X \parr \mathbb{P} \arrow[d, "g \parr \mathbb{P}"] \\
Y \parr \mathbb{P} \arrow[rr, "Y \parr \mathbb{P} \parr \mathbf{w}"'] \arrow[dr, "Y \parr \mathbb{P}"'] & & Y \parr \mathbb{P} \parr \mathbb{P} \arrow[dl, "Y \parr \mathbf{c}"] \\
& Y \parr \mathbb{P} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \parr \mathbb{P} \arrow[r, "X \parr \mathbf{w} \parr \mathbb{P}"] \arrow[dr, "X \parr \mathbb{P}"'] & X \parr \mathbb{P} \parr \mathbb{P} \arrow[r, "f \parr \mathbb{P}"] \arrow[d, "X \parr \mathbf{c}"'] & Y \parr \mathbb{P} \parr \mathbb{P} \arrow[d, "Y \parr \mathbf{c}"] \\
 & X \parr \mathbb{P} \arrow[r, "f"'] & Y \parr \mathbb{P}
\end{tikzcd}$$

□

In particular, the set of all $\mathbb{P}$-linear maps from X to $\Bbbk$ is isomorphic with the set all $\mathbb{P}$-cliques in $X^{\perp}$. The situation is similar to the one created by the monoid, but for the fact that $\mathbb{P}$ desingularizes multiplicative neutrals. The basic duality between X and $X^{\perp}$ will produce a $\mathbb{P}$-clique in $\Bbbk$, i.e. a clique in $\mathbb{P}$. We use the notation $a \sqsubset_P X$ to say that a is a $\mathbb{P}$-clique of X.

Remark. — In terms of category theory, this is not more than the remark that the functor $\_ \parr \mathbb{P}$ is a monad and that the category $\mathbf{COH}(\mathbb{I}\mathbb{P})$ is the Kleisli category of this monad. More concretely, this is nothing but the abstract nonsense version of familiar properties of free modules over a ring.

Let us give some useful definitions :

Definition 6

Assume that a and b are $\mathbb{P}$-cliques in X and Y ; then they are cliques in $X \parr \mathbb{P}$ and $Y \parr \mathbb{P}$, to which we can apply the semantic constructions of linear logic ; in particular

1. *we can apply a $\otimes$-rule between a and b, so as to get a clique in $(X \otimes Y) \parr \mathbb{P} \parr \mathbb{P}$, and we can contract the two $\mathbb{P}$, so as to get a clique $a \otimes b$ in $(X \otimes Y) \parr \mathbb{P}$, i.e. a $\mathbb{P}$-clique in $X \otimes Y$;*
2. *we can apply a $\&$-rule between a and b, so as to get a clique $a \& b$ in $(X \& Y) \parr \mathbb{P}$, i.e. a $\mathbb{P}$-clique in $X \& Y$;*
3. *we can apply a $\oplus$-rule to a, so as to get a clique $\oplus_1(a)$ in in $(X \oplus Y) \parr \mathbb{P}$, i.e. a $\mathbb{P}$-clique in $X \oplus Y$; one defines similarly the $\mathbb{P}$ clique $\oplus_2(b)$ in $X \oplus Y$;*
4. *we can apply a !-rule*[7] *to a, so as to get a clique $!a$ in $!X \parr \mathbb{P}$, i.e. a $\mathbb{P}$-clique in $!X$;*
5. *we can apply a dereliction rule so as to get a clique $\mathbf{d}_{?Z}(a)$ in $?Z \parr \mathbb{P}$, i.e. a $\mathbb{M}\mathbb{P}$-clique in $?Z$;*
6. *if $X = \Bbbk$, then we can apply a weakening rule so as to get a clique $\mathbf{w}_{?Z}(a)$ in $?Z \parr \mathbb{P}$, i.e. a $\mathbb{M}\mathbb{P}$-clique in $?Z$;*

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[7] To be precise, we use the variant introduced in [2], which allows the context to be an arbitrary Par-monoid : in this paper spaces $?Z$ are shown to be universal Par-monoids.

7. *if* $X = ?Z \parr ?Z$*, then we can apply a contraction rule so as to get a clique* $\mathbf{c}_{?Z}(a)$ *in* $?Z \parr \mathbb{P}$*, i.e. a* $\mathbb{MP}$*-clique in* $?Z$ *;*
8. *if* $Y = X \multimap Z$*, we can apply a* Modus Ponens *between* a *and* b*, so as to get a clique in* $Z \parr \mathbb{P} \parr \mathbb{P}$*, to which we can apply a contraction ; the result is noted* $b(a)$ *and it is a* $\mathbb{P}$*-clique in* Z *;*
9. *if* $Y = X^{\perp}$*, we can see* Y *as* $X \multimap \mathbb{k}$*, and get a* $\mathbb{P}$*-clique* $\langle a, b\rangle = b(a)$ *in* $\mathbb{k}$ *(we can also see* X *as* $Y \multimap \mathbb{k}$ *and introduce* $\langle b, a\rangle = a(b)$*, but observe that* $\langle a, b\rangle = \langle b, a\rangle$*, as a consequence of the commutativity of* $\mathbf{c}$*.*

Let Φ *be a functor from coherent spaces to themselves preserving directed colimits and pull-backs*[8] *; then*

1. *if* $(f(Z) \sqsubset_P \Phi(Z))$ *is a stable family of* $\mathbb{P}$*-cliques, then applying a* $\forall$*-rule yields a clique* $\forall\Phi f$ *in* $\forall\Phi \parr \mathbb{P}$*, i.e. a* $\mathbb{P}$*-clique in* $\forall\Phi$ *;*
2. *if* a *is a* $\mathbb{P}$*-clique in some* $\Phi(Z)$ *(where* Z *is a coherent space), we can apply a* $\exists$*-rule to* a*, so as to get a clique* $\exists\Phi a$ *in* $\exists\Phi \parr \mathbb{P}$*, i.e. a* $\mathbb{P}$*-clique in* $\exists\Phi$*.*

2.5 The category COH(MP)

Let us specify both a $\mathbb{M}$ and a $\mathbb{P}$; then we can speak of a $\mathbb{MP}$-clique in X as being a pair (m, a) of $m \in \mathbb{M}$ and $a \sqsubset X \parr \mathbb{P}$. $\mathbb{MP}$ morphisms from X to Y must map $\mathbb{MP}$-cliques to $\mathbb{MP}$-cliques, preserve coherent sums of non-empty families, scalar multiplication and contraction. The phenomenons already observed occur, namely that $\mathbb{MP}$-linear maps from X to Y are in natural bijection with $\mathbb{MP}$-cliques in $X \multimap Y$. In particular, the set of all $\mathbb{MP}$-linear maps from X to $\mathbb{k}$ is isomorphic to the set of all $\mathbb{MP}$-cliques in $X^{\perp}$. The $\mathbb{MP}$-cliques of $\mathbb{k}$ are pairs (m, a) of a point of $\mathbb{M}$ and a clique of $\mathbb{P}$.
The operations defined on $\mathbb{P}$-cliques extend to $\mathbb{MP}$-cliques :

Definition 7
Assume that (m, a) *and* (n, b) *are* $\mathbb{MP}$*-cliques in* X *and* Y *; then we define :*

1. $(m, a) \otimes (n, b) = (mn, a \otimes b)$ *;*
2. *if* $m = n$*,* $(m, a) \,\&\, (m, b) = (m, a \,\&\, b)$ *;*
3. $\oplus_1(m, a) = (m, \oplus_1(a))$ *;* $\oplus_2(m, a) = (m, \oplus_2(a))$ *;*
4. $!(m, a) = (m, !a)$ *;*
5. $\mathbf{d}_{?Z}(m, a) = (m, \mathbf{d}_{?Z}(a))$ *;*
6. $\mathbf{w}_{?Z}(m, a) = (m, \mathbf{w}_{?Z}(a))$ *;*
7. $\mathbf{c}_{?Z}(m, a) = (m, \mathbf{c}_{?Z}(a))$ *;*
8. $(n, b)((m, a)) = (m.n, b(a))$
9. $\langle (m, a), (n, b)\rangle = (m.n, \langle a, b\rangle)$ *;*

---

[8] The interpretation of second order quantifiers in coherent spaces is treated in appendix.

10.$\forall\Phi(m, f) = (m, \forall\Phi f)$;

11.$\exists\Phi(m, a) = (m, \exists\Phi a)$.

We shall use the notation $a \sqsubset_{MP} X$ to speak of a $\mathbb{MP}$-clique a of X.

3 Soundness

In this section we fix once for all a monoid $\mathbb{M}$, a Par-monoid $\mathbb{P}$ and a set $\mathbb{B}$ of $\mathbb{MP}$-cliques in $\Bbbk$, i.e. a set of pairs (m, a), with $m \in \mathbb{M}$ and $a \sqsubset \mathbb{P}$.

3.1 Denotational facts

Definition 8

When $a \sqsubset_{MP} X$, $b \sqsubset_{MP} X^{\perp}$, *the orthogonality relation is defined by* $a \perp b \iff \langle a, b\rangle \in \mathbb{B}$.

The commutativity of $\mathbf{c}$ implies a symmetry of orthogonality : it is not affected by the exchange of X and $X^{\perp}$, i.e. $b \perp a \iff a \perp b$: this is why we can use the double orthogonality without thinking twice.

Definition 9

A denotational fact $(X, \mathcal{F})$ *consists of a coherent space* X *together with a set* $\mathcal{F}$ *of* $\mathbb{MP}$*-cliques of* X *such that* $\mathcal{F} = \mathcal{F}^{\perp\perp}$. *When the context is clear we may say « let* $\mathcal{F}$ *be a denotational fact. ».*

3.2 Interpretation of formulas

We concentrate on the interpretation of second-order propositional linear logic. We assign to any formula A a denotational fact $(A^*, A^\bullet)$, provided we are given basic assignments $(\alpha^*, \alpha^\bullet)$ to the free variables of A : if we want to stress the influence of the interpretation $(X, \mathcal{F})$ of some variable α, we may use the more precise notation $A^*[X/\alpha], A^\bullet[\mathcal{F}/\alpha]$ (in fact A^* only depends on X, whereas $A^\bullet$ depends on both X and $\mathcal{F}$). The definition is by induction on A :

1. α^* and $\alpha^\bullet$ are part of the data ;
2. $(\alpha^{\perp})^* = (\alpha^*)^{\perp}$, $(\alpha^{\perp})^\bullet = (\alpha^\bullet)^{\perp}$;
3. $\perp^* = \Bbbk$, $\perp^\bullet = \mathbb{B}$;
4. $1^* = \Bbbk$, $1^\bullet = (\{(1, \mathbf{w}(\Bbbk))\})^{\perp\perp}$ (we use the notation $\Bbbk$ for the singleton clique of the coherent space $\Bbbk$) ;
5. $\top^* = \mathbf{0}$ (the empty coherent space), $\top^\bullet = \mathbf{0}$, i.e. it consists in all $\mathbb{MP}$-cliques in $\mathbf{0}$, i.e. all $(m, \emptyset)$;

6. $0^* = \mathbf{0}$, $0^\bullet = \emptyset^{\perp\perp}$;

7. $(A \otimes B)^* = A^* \otimes B^*$, $(A \otimes B)^\bullet = \{a \otimes b \ ; a \in A^\bullet, b \in B^\bullet\}^{\perp\perp}$;

8. $(A \parr B)^* = A^* \parr B^*$, $(A \parr B)^\bullet = \{c ; \forall a \in (A^\bullet)^\perp \quad c(a) \in B^\bullet\}$;

9. $(A \oplus B)^* = A^* \oplus B^*$, $(A \oplus B)^\bullet = (\{\oplus_1(a) \ ; a \in A^\bullet\} \cup \{\oplus_2(b) \ ; b \in B^\bullet\})^{\perp\perp}$;

10. $(A \,\&\, B)^* = A^* \,\&\, B^*$, $(A \,\&\, B)^\bullet = \{(m,a) \,\&\, (m,b) \ ; (m,a) \in A^\bullet \wedge (m,b) \in B^\bullet\}$;

11. $(!A)^* = !A^*$, $(!A)^\bullet = (\{!(m,a) \ ; m \in \mathcal{I} \wedge (m,a) \in A^\bullet)^{\perp\perp}$, where $\mathcal{I}$ is the set of $m \in \mathbb{M}$ such that $m = m^2$ and $(m, \mathbf{w}(\mathbb{k})) \in 1^\bullet$;

12. $(?A)^* = ?A^*$, $(?A)^\bullet = (\{!(m,a) \ ; m \in \mathcal{I} \wedge (m,a) \in (A^\bullet)^\perp)^\perp$;

13. $(\exists \alpha A)^* = \exists \Phi$, where Φ is the functor associating to X (taken as α^*) the coherent space $A^*[X/\alpha]$; $(\exists \alpha A)^\bullet = \{\exists \Phi(m,a) \ ; \exists (Z, \mathcal{F})) \quad (m,a) \in A^\bullet[\mathcal{F}/\alpha]\}^{\perp\perp}$ (observe the quantification over all denotational facts $(Z, \mathcal{F})$) ;

14. $(\forall \alpha A)^* = \forall \Phi$, where Φ is the functor associating to X (taken as α^*) the coherent space $A^*[X/\alpha]$; $(\forall \alpha A)^\bullet = \{(m,a) ; \forall (X, \mathcal{F})(m, a\{X\}) \in A^\bullet[\mathcal{F}/\alpha]\}$ (the notation $a\{X\}$ is explained in appendix).

Proposition 4

The interpretation of formulas is compatible with negation, i.e. $(A^\bullet)^\perp = (A^\perp)^\bullet$ *; in particular* $A^\bullet$ *is a denotational fact.*

PROOF. — The last statement is just the remark that, since $A^\bullet = (A^{\perp\bullet})^\perp$, $A^\bullet = A^{\bullet\perp\perp}$. The main statement is proved by induction on A ; some cases ($\alpha/\alpha^\perp$, $!/?$) are defined by duality and are therefore trivial. Let us check the other ones :

1. $\mathbb{B} = \{(1, \mathbf{w}(\mathbb{k}))\}^\perp$ (immediate).

2. $\mathbf{0} = \emptyset^\perp$ (immediate).

3. $(A \,\&\, B)^\bullet = ((A^\perp \oplus B^\perp)^\bullet)^\perp$; the result follows from the remark that if $(m, a \,\&\, b) \in (A \,\&\, B)^*$ and $a' \in A^{\perp *}$, $b' \in B^{\perp *}$, then $\langle (m, a \,\&\, b), \oplus_1(a') \rangle = \langle (m,a), a' \rangle$ and $\langle (m, a \,\&\, b), \oplus_2(b') \rangle = \langle (m,b), b' \rangle$.

4. $(A \parr B)^\bullet = \{a \otimes b \ ; a \in A^{\bullet\perp} \wedge b \in B^{\bullet\perp}\}^\perp$; the result follows from the remark that, if $c \in (A \parr B)^*$ and $a \in A^{\perp *}$, $b \in B^{\perp *}$, then $\langle c(a), b \rangle = \langle c, a \otimes b \rangle$.

5. $(\forall \alpha A)^\bullet = (\exists \alpha A^{\perp\bullet})^\bullet$; the result follows from the remark that, if $(\forall \Phi f) \in (\forall \alpha A)^*$ and $a \in (A^\perp[Z/\alpha])^*$ then $\langle \forall \Phi f, \exists \Phi a \rangle = \langle f(Z), a \rangle$.

□

Remark. — Let us come back to the case of the connective $\parr$. $(A \parr B)^\bullet$ can be seen as the set of $\mathbb{MP}$-linear functions c sending $A^{\perp\bullet}$ to $B^\bullet$ but also as the set of such $\mathbb{MP}$-linear maps c whose adjoint $\tilde{c}$ sends $B^{\perp\bullet}$ to $A^\bullet$. With the latter definition it amounts to verifying a condition $\langle \tilde{c}(b), a \rangle \in \mathbb{B}$, for all $b \in B^\bullet$ and all $a \in \mathcal{F}$ where $\mathcal{F}$ is such that $A^\bullet = \mathcal{F}^\perp$ (such a set is called a *preorthogonal* of $A^\bullet$). But the equation

$\langle \tilde{c}(b), a\rangle = \langle c, a \otimes b\rangle = \langle c(a), b\rangle$ shows indeed that it is enough to require that c sends a preorthogonal $\mathcal{F}$ of $A^\bullet$ into $B^\bullet$. This is indeed the most precious lemma for soundness : typically the axiom $\vdash \top, B$ will be shown to be sound on the sole grounds that $\top^\bullet$ has an empty preorthogonal.

The interpretation of formulas is easily transformed into an interpretation of sequents, i.e. $(\vdash A_1, \ldots, A_n)^\bullet$ is the same as $(A_1 \parr \ldots \parr A_n)^\bullet$ (if we take seriously our convention saying that $\parr$ is associative, there is no need to make any distinction). From the previous remark, there is the possibility to focus our definition on a particular A_k : let $\mathcal{F}_i$ be preorthogonals of the $A_i{}^\bullet$ and $a_i \in \mathcal{F}_i$ $(i \neq k)$, then the result of $n-1$ cuts, together with $n-1$ contractions, yields a member of A_k. The possibility of focusing renders the verification of soundness almost trivial, because it produces a kind of « annihilation of contexts », which reduces each rule to its simplest expression, e.g. the $\otimes$-rule is reduced to « from $\vdash A$ and $\vdash B$ conclude $\vdash A \otimes B$ ».

3.3 The soundness theorem

Definition 10

If $a \sqsubset X$, we define the $\mathbb{MP}$-clique $\iota(a) \sqsubset_{MP} X$ as $\iota(a) = (1, (X \parr \mathbf{w})(a))$. A $\mathbb{MP}$-clique of X is said to be essential *when it is of the form $\iota(a)$. Essential cliques are therefore those which do not make use of $\mathbb{MP}$, or if one prefers, which are parametric in $\mathbb{MP}$. We use the notation $a \sqsubset_e X$ to say that a is an essential $\mathbb{MP}$-clique of X.*

The familiar interpretation of the rules of linear logic (see appendix for quantifiers) assigns a clique $\pi^* \sqsubset A^*$ to a proof π of A.

Theorem 1

If π is a proof of a sequent $\vdash \Gamma$, then $\iota(\pi^*) \in \Gamma^\bullet$.

PROOF. — The proof offers no difficulty. Let us treat some cases :

1. The identity axiom $\vdash A^\perp, A$, is shown to be sound by focusing on A : if $a \in A^\bullet$, then $\iota(\pi^*)(a) = a \in A^\bullet$.

2. The exchange rule is shown to be sound essentially because of the commutativity of $\mathbf{c}$, which is already at work in our definitions.

3. The cut-rule is the first example of focusing, i.e. annihilation of the context. The rule leads from $\vdash \Gamma, A$ and $\vdash \Delta, A^\perp$ to Γ, Δ, and annihilation of contexts reduces it to a rule leading from $\vdash A$ and $\vdash A^\perp$ to $\vdash$. Soundness of the rule is nothing but checking that, if $a \in A^\bullet$, $b \in A^{\perp\bullet}$, then $\langle a, b\rangle \in \mathbb{B}$. Of course what is implicitly used is that « annihilation of contexts » commutes with the rule.

4. The case of a $\parr$-rule : by annihilation of contexts, it is reduced to « from $\vdash A, B$, deduce $\vdash A \parr B$ » ; now, the soundness is almost tautological.

5. Most other cases are as tautological, using the definition of $A^\bullet$. Only three rules deserve a closer look, namely the rules for ? : this is because this connective is defined by duality.

6. The case of a dereliction rule reduces to « from $\vdash A$ deduce $\vdash ?A$ » ; soundness of the rule is consequence of the equation $\langle \mathbf{d}_{?Z}(m,a),(n,!b)\rangle = (m.n,\langle a,b\rangle)$;

7. The case of a weakening rule reduces to « from $\vdash$ deduce $\vdash ?A$ » ; soundness of the rule is consequence of the equation $\langle \mathbf{w}_{?Z}(m,a),(n,!b)\rangle = (m.n,a)$, and that, if $n \in \mathcal{I}$, and $(m,a) \in \mathbb{B}$, then $(m.n,a) \in \mathbb{B}$;

8. The case of a contraction rule reduces to « from $\vdash ?A, ?A$ deduce $\vdash ?A$ » ; soundness of the rule is consequence of the equation $\langle \mathbf{c}_{?Z}(m,a),(n,!b)\rangle = (m.n,\langle a(!b),!b\rangle)$, and that, if $n \in \mathcal{I}$, then $n^2 = n$.

□

4 Completeness

We now restrict our attention to Π^1-formulas, i.e. formulas without second-order existential quantifiers. Let A be a closed Π^1-formula, and let a be a clique in A^* such that $\iota(a) \in A^\bullet$ for all possible $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$; then we shall establish that $a = \pi^*$ for some proof π of A. For this it will be enough to produce specific $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$, such that, if $\iota(a) \in A^\bullet$ for these particular $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$, then a is a π^*. The delicate question is indeed the commutativity of Par-monoid, and this problem will force us to replace formulas with occurrences.

4.1 The monoid

Definition 11

Let us fix once for all a denumerable set $\mathcal{L}$ of labels. *An* occurrence *is a pair (A,l) of a Π^1 formula A, and a label $l \in A$.*

The monoid $\mathbb{M}$ is made of contexts, i.e. finite multisets of formulas, but the multiplicity of formulas $?A$ is not taken into account. Multiplication is the sum of multisets and the unit is the empty multiset. An element of $\mathbb{M}$ is pure *when it is a set, i.e. all multiplicities are* 1.

Remark. — The use of occurrences is an easy way to distinguish denumerably many variants of the same formula. We shall therefore keep on speaking of formulas, but we have to remember in some cases to be precise about the labels. Typically $?A.?A = ?A$, but this idempotency only holds between the same occurrences, i.e. $(?A,l).(?A,l) = (?A,l)$. No cancellation occurs between different occurrences.

4.2 The Par-monoid

Definition 12

The Rado graph is a denumerable coherent space R with the following property : given any finite coherent spaces X and Y and embeddings[9] *f of X into Y, g of X into R, there is an embedding h of Y into R such that $g = hf$:*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\
& R &
\end{tikzcd}$$

Remark. — The Rado graph is therefore solution of a weak universal problem ; in particular it is easily shown to be unique up to isomorphism, isomorphic to its negation etc. but the isomorphisms are never unique. The Rado graph is used to interpret variables, i.e. $\alpha^* = R$: in what follows, when we refer to A^*, we shall always assume that the free variables of A have been interpreted by R.

Definition 13

The Par-monoid $\mathbb{P}$ is defined as follows :

- *$|\mathbb{P}|$ consists of all finite multisets*[10] *$\prod(A_i, x_i)$, with $x_i \in |A_i{}^*|$; such a multiset is* pure *when all the A_i are distinct ; we define $\|\prod(A_i, x_i)\| = \prod A_i$.*
- *Let $\prod(A_i, x_i)$, $\prod(B_j, y_j)$ be distinct elements of $|\mathbb{P}|$; then $\prod(A_i, x_i) \frown \prod(B_j, y_j)$ iff*
  - *both are pure*
  - *$\|\prod(A_i, x_i)\| = \|\prod(B_j, y_j)\|$, hence by purity $\prod(B_j, y_j) = \prod(A_i, x'_i)$, for a unique family x'_i*
  - *$x_i \frown x'_i$ for some index i.*
- ***w*** *(seen as a clique) is the singleton $\{1\}$, where 1 is the empty product ;*
- ***c*** *is given by its trace, which consists of 3-tuple s of elements of $|\mathbb{P}|$:*
  - *all $(a, 0, a)$;*
  - *all $(0, a, a)$;*
  - *all (a, b, c) where a, b, c are pure, with $\|a\| = \|b\| = \|c\| = ?\Gamma$ for some Γ, i.e. $a = \prod(?A_i, x_i)$, $b = \prod(?A_i, y_i)$, $c = \prod(?A_i, z_i)$ and $(x_i, y_i, z_i) \in \mathbf{c}_{?A_i{}^*}$; here we use the structure of Par-monoid of $?A_i{}^*$.*
  - *more generally, all $(a.a', b.b', c.a'.b')$, with $a.a', b.b'$ pure, $\|a'\| \cap \|b'\| = \emptyset$, and (a, b, c) is like in the previous case.*

---

[9] See definition 16.

[10] We use a multiplicative notation for multisets ; hence the empty multiset is noted 1 and the usual sum of multisets is a product.

Proposition 5
$\mathbb{P}$ is a Par-monoid.

Proof. — The proof basically uses the fact that $?{A_i}^*$ is a Par-monoid. Observe that $\mathbf{c}$ never contains (a, b, c) when $\|a\| = \|b\|$ is a formula (A, l) not beginning with an interrogation mark : it would be impossible to ensure commutativity. In compensation we have 3-tuple s (a, b, c) when $\|a\| = (A, l)$, $\|b\| = (A, l')$ and $l \neq l'$. □

4.3 The denotational fact $\mathbb{B}$

Let us slightly modify the rules of linear logic, in order to cope with occurrences : sequents must now be pure, i.e. without repetitions : if we want to use A twice in Γ, we must use two different occurrences (A, l) and (A, l'). This induces another modification of the rules, i.e. we can no longer write an explicit contraction rule, since we would be forced to have twice the same occurrence in premise. So we drop contraction, but we allow in the two multiplicative binary rules ($\otimes$ and CUT) some formulas $(?C, l)$ to be present in both premises, although the conclusion retains only one copy of any such formula.

Definition 14
$\mathbb{B}$ consists of all $\Gamma.\pi^* = \Gamma.\{\prod(A_i, x_i) \ ; \prod A_i = \Gamma, (x_i) \in \pi^*\}$, such that π is a proof of Γ (so that Γ is pure) and of all $\Gamma.a$ such that Γ is impure.

$\mathbb{B}$ has some immediate properties which are easily transferred to any denotational fact :

Proposition 6
Let $\mathcal{F}$ be a denotational fact ; then

1. *All $\Gamma.a$ with Γ impure belong to $\mathcal{F}$. In the next cases, we assume that labels have been chosen so as to ensure that all sequents involved are pure.*
2. *If $((\Gamma, A, B).a) \in \mathcal{F}$ and a' is obtained from a by applying a $⅋$-rule between A and B, then $((\Gamma, A ⅋ B).a') \in \mathcal{F}$.*
3. *Denotational facts are closed under dereliction : if $((\Gamma, A).a) \in \mathcal{F}$ and a' is obtained by a dereliction on A, then $((\Gamma, ?A).a') \in \mathcal{F}$.*
4. *Denotational facts are closed under weakening : if $\Gamma.a \in \mathcal{F}$ and a' is obtained by a weakening on $?A$, then $((\Gamma, ?A).a') \in \mathcal{F}$.*

Proof. — More or less immediate : each property is shown to hold for $\mathbb{B}$, then it is transferred to any denotational fact by orthogonality. □

Proposition 7
The set $\mathcal{I}$ consists of all contexts $?\Gamma$.

PROOF. — The only idempotents are the $?\Gamma$; moreover the closure of denotational facts under weakening forces $?\Gamma.\mathbf{w}(\Bbbk)) \in 1^\bullet$. $\square$

4.4 The completeness theorem

Let us add a new label l_0 to $\mathcal{L}$, so as to ensure that, when we speak of $\vdash A, \Gamma$, we can give this extra label to A. A proof of $\vdash A, \Gamma$ induces a clique in $A^* \parr \Gamma^*$, and therefore as a clique in $A^* \parr \mathbb{P}$, i.e. a $\mathbb{P}$-clique in A^*.

Definition 15
Given any Π^1 formula A, we introduce

$$A^\circ = \{\Gamma.a\ ; \Gamma\ \text{ impure } \vee \exists\pi(\pi\ \text{ proof of }\ \vdash A, \Gamma\} \wedge a = \pi^*)$$

(the definition uses the new label facility). From this we can define a model by means of $\alpha^\bullet = \alpha^\circ$.

Proposition 8
If A is Π^1, then $A^\bullet \subset A^\circ$.

PROOF. — Observe that A° is a denotational fact : if Γ is pure, then $\Gamma.a \in A^\circ$ iff $a = \pi^*$ where π is a proof of $\vdash \Gamma, (A, l_0)$. Using the $\mathbb{P}$-clique id_l^A of $A^{\perp *}$ which interprets the identity axiom $\vdash (A^\perp, l_0), (A, l)$, we can observe that $\langle a, id_l^A\rangle$ is nothing but the interpretation of π, when l_0 has been replaced with l ; moreover, l can be chosen so as to make $\Gamma, (A, l)$ pure. We easily conclude that $\Gamma.a \in A^\circ$ iff it is orthogonal to all cliques id_l, i.e. A° is the orthogonal of something. This observation is crucial, since we no longer need to check the inclusion $A^\bullet \subset A^\circ$: we can content ourselves with $\mathcal{F} \subset A^\circ$, where $A^\bullet = \mathcal{F}^{\perp\perp}$.
The proof is rather trivial, and is by induction on A ;

1. $\alpha^\bullet \subset \alpha^\circ$ by definition.
2. $\alpha^{\perp\bullet} \subset \alpha^{\circ\perp}$; assume that $\Gamma.a \perp \alpha^\circ$ and that Γ is pure. But $id_l^{\alpha^\perp} \in \alpha^\circ$, and since $\alpha^{\perp\circ} = \{id_l^{\alpha^\perp}; l \in L\}^\perp\}$, we easily conclude that $\Gamma.a \in \alpha^{\perp\circ}$.
3. $1^\bullet = \{1.\mathbf{w}(\Bbbk))\}^{\perp\perp}$, but $\Bbbk$ is the interpretation of the axiom for 1, hence $1^\bullet \subset 1^{\circ\perp\perp} = 1^\circ$.
4. $\perp^\bullet = \mathbb{B} = \perp^\circ$.
5. Since $0^\bullet = \emptyset^{\perp\perp}$, $0^\bullet \subset 0^\circ = 0^{\circ\perp\perp}$.
6. $\top^\circ$ consists of all $\Gamma.a$ such that, if Γ is pure, then $\vdash \top, \Gamma$ is provable and a is the interpretation of such a proof, which must be an empty clique, i.e. $\top^\circ = \top^\bullet$.

7. Assume that $A^\bullet \subset A^\circ$, $B^\bullet \subset B^\circ$; $(A \otimes B)^\bullet = \mathcal{F}^{\perp\perp}$, with $\mathcal{F} = \{a \otimes b \; ; a \in A^\bullet, b \in B^\bullet\}$ (indeed we can even restrict to $\mathcal{G} = \{\Gamma.c \in \mathcal{F}; \Gamma \text{ pure}\}$). If $a \in A^\bullet$, $b \in B^\bullet$, both come from proofs, and by a $\otimes$-rule $a \otimes b \in (A \otimes B)^\circ$, at least when the restrictions on purity apply, which shows that $\mathcal{G} \subset A \otimes B^\circ$, and so $(A \otimes B)^\bullet \subset (A \otimes B)^{\circ\perp\perp} = (A \otimes B)^\circ$.

8. Assume that $A^\bullet \subset A^\circ$, $B^\bullet \subset B^\circ$; this implies that $(A, l).id_l^A \in A^{\perp\bullet}$, $(B, l').id_{l'}^B \in B^{\perp\bullet}$, so $(((A, l), (B, l')).id_l^A \otimes id_{l'}^B) \in A^{\perp\bullet} \otimes B^{\perp\bullet} \subset (A \parr B)^{\bullet\perp}$. Now we observed that denotational facts are closed under the $\parr$-rule, and applying this to $id_l^A \otimes id_{l'}^B$, we get $(A \parr B, l").id_{l"}^{A \parr B} \in (A \parr B)^{\bullet\perp}$, hence $(A \parr B)^\bullet \subset (A \parr B)^\circ$.

9. Assume that $A^\bullet \subset A^\circ$, $B^\bullet \subset B^\circ$; $(A \oplus B)^\bullet = \mathcal{F}^{\perp\perp}$, with $\mathcal{F} = \{\oplus_1(a) \; ; a \in A^\bullet\} \cup \{\oplus_2(b) \; ; b \in B^\bullet\}$. If $\Gamma.a \in A^\bullet$, and Γ is pure, then a comes from a proof of Γ, A, and $\oplus_1(a)$ comes from a proof of $\Gamma, A \oplus B$; similarly if $\Gamma.b \in B^\bullet$, then $\oplus_2(b)$ comes from a proof of $\Gamma, A \oplus B$. From this we get $\mathcal{F} \subset (A \oplus B)^\circ$, hence $(A \oplus B)^\bullet \subset (a \oplus B)^{\circ\perp\perp} = (A \oplus B)^\circ$.

10. Assume that $A^\bullet \subset A^\circ$, $B^\bullet \subset B^\circ$; if $\Gamma.(a \,\&\, b) \in (A \,\&\, B)^\bullet$, and Γ is pure, then $\Gamma.a \in A^\bullet$, hence it comes from a proof of $\vdash \Gamma, A$; for similar reasons b comes from a proof of $\vdash \Gamma, B$, and applying a &-rule, we see that $a \,\&\, b$ comes from a proof of $\vdash \Gamma, A \,\&\, B$; this proves that $(A \,\&\, B)^\bullet \subset (A \,\&\, B)^\circ$.

11. Assume that $A^\bullet \subset A^\circ$; then $!A^\bullet = \mathcal{F}^{\perp\perp}$, with $\mathcal{F} = \{\Gamma.!a; \Gamma \in \mathcal{I} \wedge a \in A^\bullet\}$. Now, $\mathcal{I}$ consists exactly of all contexts $?\Gamma$. If $a \in \mathcal{F}$, then a comes from a proof of $?\Gamma, a$, and $!a$ corresponds to the application of a !-rule to this proof, i.e. $!a$ comes from a proof of $?\Gamma, !A$. From this we get $\mathcal{F} \subset !A^\circ$, hence $!A^\bullet \subset !A^{\circ\perp\perp} = !A^\circ$.

12. Assume that $A^\bullet \subset A^\circ$; then $(A, l).id_l^A \in A^{\perp\bullet}$, and by closure of denotational facts under dereliction, we get another clique a such that $(?A, l').a \in A^{\perp\bullet}$. From this $(?A, l').!a \in !A^{\perp\bullet}$, i.e. $(?A, l').id_l^{?A} \in !A^{\perp\bullet}$, from which we get $?A^\bullet \subset ?A^\circ$.

13. Assume that $A^\bullet \subset A^\circ$; then, if $\Gamma.a \in (\forall\alpha A)^\bullet$, we get in particular that $\Gamma.a\{R\} \in A^\bullet \sqsubset A^\circ$. This forces $a\{R\}$ to be the interpretation of a proof of $\vdash \Gamma, A$. Since R is big enough, $a\{R\}$ defines a uniquely, hence, if we apply a $\forall$-rule to the function $a\{X\}$, we get the interpretation of a proof of $\vdash \Gamma, \forall\alpha A$. So $(\forall\alpha A)^\bullet \subset (\forall\alpha A)^\circ$.

□

Theorem 2

If A is a closed Π^1 formula and $\iota(a)$ is an essential clique in $A^\bullet$, then $a = \pi^*$, for some proof π of A.

Proof. — We apply the proposition and conclude that $\iota(a) \in A^\circ$, and since the empty context is pure, we conclude that $\iota(a)$ is the interpretation of a proof of A in the empty context, i.e. that a is the interpretation of a proof of A. □

A The semantics of second-order quantifiers

We sketch the interpretation of second-order quantifiers. The original interpretation is to be found in [1], for the intuitionistic case, i.e. system $\mathbb{F}$. We stick to the notations introduced in [3].

A.1 Stable functors

Definition 16

> *If X and Y are coherent spaces, an* embedding *from X to Y is an injective map f from the web $|X|$ of X to the web $|Y|$ of Y such that* $\forall x, y \in |X| \quad (x \frown y \Leftrightarrow f(x) \frown f(y))$. *An embedding induces a linear map (still denoted f) from X to Y : $f(a) = \{f(x)\ ; x \in |X|\}$ and a linear map f^{-1} from Y to X : $f^{-1}(b) = \{x \in |X|\ ; f(x) \in b\}$. Obviously $f^{-1}f = id^X$, whereas ff^{-1} is idempotent, i.e. is a projection of Y.*
> *The category $\mathbf{COH_e}$ is defined as the category of coherent spaces equipped with embeddings (instead of linear maps). We denote by $\Im(X, Y)$ the set of all embeddings from X into Y.*

Proposition 9

> *In $\mathbf{COH_e}$ any coherent space is the directed colimit of finite coherent spaces.*

PROOF. — Let X be a coherent space, and let I consist of all finite subsets of $|X|$. If $i \in I$, then $|X| \cap i$ is naturally equipped with a structure of coherent space X_i, and X is the union of the X_i, indeed a directed union. Inclusion maps are the simplest embeddings, and directed colimits of inclusions are exactly directed unions. □

All connectives of linear logic are easily transformed into (covariant) functors. Assume that $f \in \Im(X, X')$ and $g \in \Im(Y, Y')$; then

1. one defines $f \otimes g \in \Im(X \otimes Y, X' \otimes Y')$ by $(f \otimes g)(x, y) = (f(x), g(y))$;
2. one defines $f \parr g \in \Im(X \parr Y, X' \parr Y')$ by $(f \parr g)(x, y) = (f(x), g(y))$;
3. one defines $f \,\&\, g \in \Im(X \,\&\, Y, X' \,\&\, Y')$ by $(f \,\&\, g)(x, 0) = (f(x), 0)$, $(f \,\&\, g)(y, 1) = (g(y), 1)$;
4. one defines $f \oplus g \in \Im(X \oplus Y, X' \oplus Y')$ by $(f \oplus g)(x, 0) = (f(x), 0)$, $(f \oplus g)(y, 1) = (g(y), 1)$;
5. one defines $!f \in \Im(!X, !X')$ by $!f([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)]$;
6. one defines $?f \in \Im(?X, ?X')$ by $?f([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)]$.

Remark. — In fact these functors are nothing but existing constructions : for instance $f \otimes g$ is just the interpretation of the derived rule « From $X \multimap Y$ and $X' \multimap Y'$ conclude $X \otimes X' \multimap Y \otimes Y'$ », restricted to the case where the premises are interpreted by embeddings. But this does not apply to negation (which is not an official connective in our setting), since the rule « From $X \multimap Y$ deduce $X^\perp \multimap Y^\perp$ » is wrong. But the crucial point of the restriction to embeddings is that negation becomes a covariant functor : just define $f^\perp = f$. Moreover it is obvious from the definitions that our functors are compatible with negation, e.g. $(f \otimes g)^\perp = f^\perp \parr g^\perp$.

Proposition 10

The functors corresponding to the connectives of linear logic preserve directed colimits and pull-backs.

Proof. — An embedding is the composition of an inclusion map with an isomorphism. All our functors preserve inclusions. Preservation of direct limits is basically nothing but the fact that all our functors preserve directed unions, which is immediate from the finitary character of our constructions (directedness is only needed for exponentials). The pull-back between two inclusions $X \subset Z$ and $Y \subset Z$ is noting but the inclusion $X \cap Y \subset Z$; preservation of pull-backs is therefore nothing more than preservation of intersections. □

Remark. — Any first order formula A can be seen as a functor of its variables. More precisely to any formula A we can associate a functor from $\mathbf{COH_e}^n$ to $\mathbf{COH_e}$, where the integer n is used to index the free variables of A. Basically $A^*(X_1, \ldots, X_n)$ is the interpretation of A when $\alpha_1, \ldots \alpha_n$ are respectively interpreted by $X_1, \ldots, X_n$, and $A^*(f_1, \ldots, f_n) \in \Im(A^*(X_1, \ldots, X_n), A^*(Y_1, \ldots, Y_n))$ when the $f_i \in \Im(X_i, Y_i)$. These functors preserve directed colimits and pull-backs ; indeed, the interpretation (to be defined) of second-order quantifiers will still produce functors preserving directed colimits and pull-backs, i.e. the general paradigm of dependency of a formula over a second-order variable is that of a functor from $\mathbf{COH_e}$ to $\mathbf{COH_e}$ preserving directed colimits and pull-backs.

Definition 17

A stable *functor is a functor from* $\mathbf{COH_e}$ *to itself preserving direct limits and pull-backs.*

Proposition 11

Let Φ be a stable functor from $\mathbf{COH_e}$ *to itself. Let X be a coherent space, and let $z \in |\Phi(X)|$; then*

1. *There exits a finite coherent space X_0 together with an embedding $f \in \Im(X_0, X)$ and $z_0 \in |\Phi(X_0)|$ such that $z = \Phi(f)(z_0)$; in the sequel, X_0 is chosen with the smallest possible cardinality.*
2. *Assume that X_1, g, z_1 are such that $z_1 \in |\Phi(X_1)|$ and $z = \Phi(g)(z_1)$, then there exists an embedding $h \in \Im(X_0, X_1)$ such that $f = gh$ and $z_1 = \Phi(h)(z_0)$; moreover, if $h' \in \Im(X_0, X_1)$ is such that $f = gh'$ and $z_1 = \Phi(h')(z_0)$, then h and h' have the same range.*

PROOF. — Nothing essential is lost if we assume that Φ actually preserves inclusions. Then $|\Phi(X)| = \bigcup \Phi(X_i)$ for a directed family (X_i) of finite coherent spaces, and we can find some index (let us say 0) such that $z \in \Phi(X_0)$. We are done with the first part of the proposition. Let us therefore choose X_0 minimal among those coherent spaces which are *included* in X such that $z \in \Phi(X_0)$ (and for f the inclusion map). We claim that X_0 is uniquely determined by this requirement, for if X_1 is any sub-coherent space of X such that $z \in \Phi(X_1)$, then by preservation of intersections, $z \in \Phi(X_0 \cap X_1)$, which forces $X_0 \subset X_1$. This almost proves the second statement, taking for h the inclusion between X_0 and X_1. Now observe that the range of h is exactly X_0, and if we were taking another solution h' (maybe no longer an inclusion) this would still be true. □

Remark. — In the second statement, h is not unique, which would be the case if Φ were preserving kernels (i.e. equalizers). What may happen is that X_0 has a non-trivial automorphism σ such that $\Phi(\sigma)(z) = z$. The simplest case is with $\Phi(X) = !X$ and X_0 has two coherent atoms a and b : if σ flips a and b, then $!\sigma([a,b]) = [a,b]$.

Let us now select, once for all a denumerable family F_i of finite coherent spaces such that :

1. Any finite coherent space is isomorphic to some F_i.

2. Any two F_i with distinct indices are non-isomorphic.

Such a family (F_i) is called a *basis* ; since we selected one basis, we shall refer to it as « the » basis.

Definition 18

If Φ is a stable functor, then we define $Tr(\Phi)$ to consist of all pairs (F, z) such that F belongs to the basis, $z \in |\Phi(F)|$ and for any embedding h from some element of the basis G into F, z is not in the range of f, unless $F = G$. On the trace we define an equivalence : $(F, z) \simeq (F', z')$ iff $F = F'$ and $z = \Phi(\sigma)(z')$ for some automorphism σ of F. In practice we work on the quotient $Tr(\Phi)/\simeq$, but we shall work as if we had chosen one (F, z) in any equivalence class.

Our proposition can now be restated as : $z \in |\Phi(X)|$ can be written $z = \Phi(f)(z_0)$ for some $(F, z_0) \in Tr(\Phi)$ and $f \in \Im(F, X)$. (F, z_0) is unique up to $\simeq$ and the range of f is uniquely determined.

A.2 Stable families

Definition 19

Let Φ be a stable functor ; then we define the coherent space $\forall\Phi$ as follows :

- ▸ *$|\forall\Phi|$ consists of all (equivalence classes) of pairs $(F, z) \in Tr(\Phi)$ such that : for all G in the basis, for all embeddings $f, f' \in \Im(F, G)$, then $\Phi(f)(z) \frown\!\!\!\!\smile \Phi(f')(z)$.*

- $(F, z) \overset{\frown}{\smile} (F', z')$ *iff for all* G *in the basis, for all embeddings* $f \in \Im(F, G)$ *and* $f' \in \Im(F', G)$*, then* $\Phi(f)(z) \overset{\frown}{\smile} \Phi(f')(z')$*.*

We define the coherent space $\exists\Phi$ *by duality :* $\exists\Phi = (\forall\Phi^{\perp})^{\perp}$*.*

Remark. — In the previous definition we can replace the quantification on G by a quantification over all coherent spaces, which is not extremely elegant. We can also take an opposite bias, namely replace the quantification on all G in the basis by a quantification over all G in the basis and all f, f' such that the union of the ranges of f and f' covers G. Such a quantification is indeed a finite one, which shows the finitary character of our definition.

Definition 20

Let Φ *be a stable functor ;* a stable family *(of cliques) in* Φ *is a family* (a_X) *indexed by all coherent spaces, such that :*

- $a_X \sqsubset \Phi(X)$
- *if* $f \in \Im(X, Y)$*, then* $a_X = \Phi(f)^{-1}(a_Y)$*.*

Remark. — A stable family is nothing more than a cartesian natural transformation from the constant functor $\mathbb{k}$ to Φ. Cartesianity, which is a pull-back diagram, is expressed by $a_X = \Phi(f)^{-1}(a_Y)$, which is stronger than $\Phi(f)(a_X) \subset a_Y$.

Proposition 12

Stable families in Φ *are in* $1 - 1$ *correspondence with cliques in* $\forall\Phi$*. More precisely :*

1. *To any coherent clique* $a = (a_X)$*, we associate its* trace $Tr(a) = \{(F, z) \in |\forall\Phi| \; ; z \in a_F\}$*.*
2. *To any clique* $A \sqsubset \forall\Phi$ *we associate a coherent clique* $(A\{X\})$*, defined by* $A\{X\} = \{\Phi(f)(z) \; ; (F, z) \in A, f \in \Im(F, X)$*.*

PROOF. — It is not difficult to check that $Tr(a)$ is a clique and that $(a\{X\})$ is a stable family. It remains to show that the two operations are reciprocal.
Let us check the most delicate point, namely that, if $a = (a_X)$, then $Tr(a)\{X\} = a_X$, whose non-trivial part is the inclusion $a_X \subset Tr(a)\{X\}$. Now, if $zX \sqsubset \Phi(X)$, then one can find $(F, z_0) \in Tr(\Phi)$ (indeed in $|\forall\Phi|$) and $f \in \Im(F, X)$ with $z = \Phi(f)(z_0)$; the property will be immediate if we can show that $(F, z_0) \in Tr(a)$. Now observe that $a_F = \Phi(f)^{-1}(a_X)$, from this we get $z = \Phi(f)(z_0')$ for some $z_0' \simeq z_0$, i.e. $z_0' = \Phi(\sigma)(z_0)$ for some automorphism σ of F, and so $z \in \Phi(\sigma)^{-1}(F) = \Phi(F)$, i.e. $(F, z) \in Tr(a)$. □

Remark. — The interpretation of quantifiers will therefore work in strict analogy to the case of implication : the trace replaces a function (stable family) with a clique, and the operation $A\{X\}$ defines a function from a clique. These two operations can therefore be taken as the interpretations of $\forall$-introduction and $\forall$-elimination, if we were dealing with natural deduction ; with our syntactical choice, $A\{X\}$ will be used for the $\exists$-rule. Should we work in system $\mathbb{F}$, we would interpret the reduction rule $(\Lambda\alpha t)\tau \rightsquigarrow t[\tau/\alpha]$, by the denotational equality $Tr(t^*)\{\tau^*\} = t_{\tau^*}$: in linear logic this denotational equality is behind the denotational invariance under cut-elimination.

A.3 Interpretation of quantifiers

If A induces a stable functor Φ (corresponding to the variable α), then we can set $(\forall\alpha A)^* = \forall\Phi$ and $(\exists\alpha A)^* = \exists\Phi$. Moreover, if Φ is indeed a stable functor of several variables, then $\forall\alpha A$ and $\exists\alpha A$ are stable functors of the remaining variables, so that we can go on and interpret any quantified formula.
It remains to interpret the logical rules for second-order quantifiers. But remember that we need to assign some arbitrary coherent space X to any free variable α of A, with the consequence that the interpretation π^* of a proof π of a formula A (in which α occurs) is a family indexed by X, indeed a stable one ; the same is true for the proof of a sequent.

Definition 21

1. *Let π be a proof of $\vdash \Gamma, A$, and assume that $\pi^* = \pi^*_X$ is the coherent family interpreting π (with $\alpha^* = X$), and let π' be obtained from π by a $\forall$-rule, with conclusion $\vdash \Gamma, \forall\alpha A$ (so α is not free in Γ). We define $(\pi')^* = \{\underline{x}(F,z) \,; \underline{x}z \in \pi^*_F\}$.*

2. *Let π be a proof of $\vdash \Gamma, A[B/\alpha]$, and let π' be obtained from π by means of an $\exists$-rule, with conclusion $\vdash \Gamma, \exists\alpha A$. We define $\pi'^* = \{\underline{x}(F,z) \,; (F,z) \in |\exists\alpha A| \wedge \exists f \in \Im(F, B^*)(\underline{x}\Phi(f)(z) \in \pi^*\}$.*

The denotational invariance under cut-elimination is essentially a consequence of proposition 12 ; denotational invariance was used in proposition 4 under the form $\langle\forall\Phi f, \exists\Phi a\rangle = \langle f(Z), a\rangle$. It would also be necessary to check that all families constructed (including those coming from the last definition) are stable ones, but this is straightforward.

B Open questions

B.1 First order quantifiers

Our result deals with second-order propositional calculus, whereas the standard completeness result for classical logic involves first-order predicate calculus (and can be extended to Π^1 formulas without any problem). So the question is to which extent one can get a denotational completeness for closed Π^1-formulas of second-order linear predicate calculus. This seems to offer no conceptual difficulty, but it may stumble on some technicalities.

- The denotational interpretation of first-order quantification is not something very natural. Should a first-order $\forall$ be seen as an infinite &, and in this case what is the index set ? Or should it be treated in a more uniform spirit ?

- When we speak of a closed formula, it means « no parameters », in particular, predicate constants and function symbols must be universally quantified ; the question of second-order function quantifiers may be technically delicate.

B.2 Non-commutative logic

Obviously the use of non-commutative $\mathbb{M}$ and $\mathbb{P}$ would model non-commutative variants of linear logic. The question is not that much the question of extending completeness, but the question of using the new idea to choose between the several known variants of non-commutative linear logic.

B.3 Why a bimodule ?

The monoid plays a modest role, and it is likely that the present structure of « bimodule » can be collapsed to something simpler. We say « simpler », and we mean it, i.e. one should exclude solutions which consist in encoding $\mathbb{M}$ into $\mathbb{P}$ but which blur the global structure.

B.4 Plain completeness

Classical completeness is « a first-order formula A true in any model is provable » ; now if we quantify A over all its parameters (domain of the model, predicate constants, function symbols) we get a Π^1-formula B, and the truth of A in any model is the same as the truth [11] of B. Hence classical completeness is exactly « A true closed Π^1-formula is provable ». There is something slightly unsatisfactory in our extension, since the classical case refers to plain truth, whereas, we have to parametrize our models by $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$. Is this in the nature of things, or is there a natural choice of $\mathbb{M}$, $\mathbb{P}$ and $\mathbb{B}$ (we exclude as a natural choice the 3-tuple constructed for the sake of the proof).

B.5 Other linear semantics

Let us end with a technical question : the use of the monoid and the Par-monoid is enough to get completeness w.r.t. coherent semantics, which was already a reasonable semantics in terms of completeness (e.g. not identifying $\otimes$ and $\parr$ etc.). Would it be the same with semantics coming from standard algebra and which are known to make « mistakes » ? In other terms, can we perform the same kind of modification on any decent semantics and get completeness ? A positive answer would be very interesting.

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[11] Please do not « correct » this into « truth in the standard model »... what do you think that *standard* means ?

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