UPENN 7 MARS 2001

FROM THE
RULES OF LOGIC
TO THE
LOGIC OF RULES

Jean-Yves Girard

FORMALISMS

En mathématiques le XX$^{\text{ème}}$ siècle commence vers 1890...

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- Physics made of islets linked by hazardous passerelles. In constrast to unity de principe of la mathématique. Analysis and algebra (calculus on letters, variables, equations) do not contradict each other.
- Central role of natural numbers ; Peano's arithmetic PA, one of the very first examples of a formal system.

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Terms : $\mathbf{0}$ $\mathbf{x, y, z, \ldots}$ $\mathbf{St}$ $\mathbf{t + t'}$ $\mathbf{t \times t'}$

`(zero, variables, successor (+1), sum, product)`

Example : $\mathrm{SSSSSS0}$ **represents** $6\ldots$ **(compare with** IIIIII**).**

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$$t = t' \quad \neg P \quad P \vee P' \quad P \wedge P' \quad P \Rightarrow P' \quad \forall x P \quad \exists x P$$

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$$\forall x \forall y \forall z (x \neq 0 \wedge y \neq 0 \wedge z \neq 0) \Rightarrow x^3 + y^3 \neq z^3$$

with $t \neq u :: \neg(t = u), t^3 :: (t \times (t \times t))$

(A case of Fermat's last theorem).

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Axioms : $P \Rightarrow P$ $x = x$ $\ldots$ (logic)

$x + 0 = x$ $x + Sy = S(x + y)$ $x \times 0 = 0$ $x \times Sy = (x \times y) + x$

$Sx \neq 0$ $Sx = Sy \Rightarrow x = y$ (arithmetic).

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(arithmetic).

Proof rules :

$$\frac{P \quad P \Rightarrow Q}{Q}$$

(Modus Ponens)

$$\frac{P[0] \quad P[x] \Rightarrow P[Sx]}{P[y]}$$

(Recurrence or Induction)

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Theorems : $SSS0 + SS0 = SSSSS0$ $\quad \forall x\ (0 + x = x)$

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```
Syntax error
A fatal error appeared at 0028:C000BCED in the
VXD VMM(01) + 0000ACED.
```

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- Negative answer :

NOT TO KNOW ≠ TO KNOW NOT

RECESSIVE ≠ EXPANSIVE

PARADOXES

Est-ce grave, Docteur ?

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Dictionary : For a given size, sorted lexicographically.

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- List of **all** infinite lists of zeros and ones ? Impossible because of **diagonal argument**. Let $L_1, L_2, L_3, \ldots$ be a list of **all** infinite lists ; dispose them one above another and then. . .

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0 0 1 1 1 0 0 1 0 0 0 1 1 1 1 0 0 0 0 1 1 1 0 . . .
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- Mathematical formalism is expansive too : it accumulates theorems. In sharp contrast with medicine.

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- **Hilbert (1900) : prove the consistency (non-contradiction) of arithmetic. Recidive around 1920 with a finitistic programme. Reduction to the sole formal paradoxes.**

FORMALISTS

Zorro est arrivé...

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- For instance one could show that a theorem has necessarily an even number of symbols ; if P is provable, $\neg P$ is « odd » hence not provable… Too naive !

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- In mathematics, verifying up to precision $1/N$ means checking up to integer N : Hilbert, Popper, même combat !
- Consistency is recessive : « so far no contradiction ».
- A property is expansive when its negation is recessive. Example « provability » vs. « consistency ».

INCOMPLETENESS

Non, c'était Gödel…

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- Properties of the formalism represented by propositions of Peano's arithmetic PA ; e.g., « P is provable » becomes $\mathrm{Thm}_{\mathrm{PA}}[\ulcorner P \urcorner]$, « PA is consistent » becomes $\mathrm{Con}_{\mathrm{PA}}$. No immediate arithmetical meaning.

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- Proposition $G \sim \neg\mathrm{Thm}_{\mathsf{PA}}[\ulcorner G \urcorner]$: « I am not provable »
  The liar's antinomy « I am lying » ?...

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- RECESSIVE $\neq$ EXPANSIVE
  TRUE $\neq$ PROUVABLE
  P NOT PROVABLE $\neq \neg\mathrm{P}$ PROVABLE.

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- Hilbert's formalism eventually dies of overscientism : to prove the consistency in mathematics !
  Vous l'avez voulu, Georges Dandin !

INTUITIONNISMS

Pendant que Dupond et Dupont progressaient hardiment…

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- Linear logic (1985) : symmetry program/environment. Procedural logic, no longer realist.

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- $\Omega \leq \mathfrak{D} \leq$ ✠. The (non)-design Ω (Faith) and the design ✠ (Daimon) as paradigms of recessivity and expansivity.

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  General case : Intersection type.

  Mystery of incarnation : When behaviours disjoint,

$$|\mathbf{G} \cap \mathbf{H}| = |\mathbf{G}| \times |\mathbf{H}|$$

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- « Millenium question » :

P VS. NP

(à suivre)

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`http://iml.univ-mrs.fr/~girard/Articles.html`