On the meaning of logical rules I : syntax vs. semantics[*]

Jean-Yves Girard
Institut de Mathématiques de Luminy, UPR 9016 – **CNRS**
163, Avenue de Luminy, Case 930, F-13288 Marseille Cedex 09

girard@iml.univ-mrs.fr

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Abstract

> « *Oui, c'est imbécile ce que je dis ! Seulement je ne sais pas comment concilier tout ça. Il est sûr que je ne me sens libre que parce que j'ai fait mes classes et que je ne sors de la fugue que parce que je la sais.* »
> Claude-Achille Debussy
> *Entretiens avec Ernest Guiraud,* ∼ 1890.

This paper is the first of a new series : ten years ago I wrote [15], a programmative paper that launched the program of *Geometry of Interaction.* This program has now found its natural limits, i.e. it must be technically and methodologically revisited because of recent progress —e.g. the localization of time in logic and the development of what is known as *Game Semantics.* This paper expresses more a maturation than a revision of the old program.
Traditionally syntax and semantics live in completely distinct worlds, one finite and accessible, the other infinite and abstract. However, the spirit of game interpretation is quite different, since proofs and refutations (tests) are represented by players which are homogeneous in nature. Therefore the old dualism is replaced with a *monism*... and we are led to consider the opposition between syntax and semantics as an obsolete legacy of the XIX$^{\text{th}}$ century : in particular the traditional logical issues of logic —completeness and soundness— should be completely revisited.
The question at stake is the question of *meaning*, and *meaning* means *meaning*, not a substitute like *use* or *interpretation.* My thesis is that the meaning of logical rules is to be found in the *well-hidden geometrical structure of the rules themselves* : typically, negation should not be interpreted by « NO », but by the exchange between *Player* and *Opponent.*
This opens a a new field of discussion : Gödel's theorem forbids in principle any internal justification, and Tarski's explanation of truth presupposes a meta-world in which the principles of logic already make sense... i.e. the educated tradition backs a *transcendental* view of logic. But when we approach the evading meaning of operations, we discover certain internal harmonies of logic which can be seen as a plea for a sort of *immanentism* more in

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[*] The main technical novelty behind this paper, i.e. the localization of time in logic was found during my stay in the Certosa of Pontignano, Siena, March-April 1997, a stay partly founded by the group G.N.S.A.G.A. of the Italian Consiglio Nazionale delle Ricerche ; I am especially grateful to Aldo Ursini who invited me.

the spirit of usual mathematics. In other terms Gödel's theorem is not the end of any foundational reflection about logic : we should only replace a theory of the *why* with a theory of the *how.*
Another way to present the program is to seek a convincing explanation of the most mysterious discovery of Gentzen, the *subformula property*, see Annex B, which is a *purely internal* completeness result (in the original sense of « Nothing is missing »). And —by the way— we could say that we interpret Gentzen by Gentzen since the game interpretation also originates in Gentzen's work... The answer (see [19] for precise details) is a universal game, in which only certain kinds of strategies, called *designs* are considered. Logic comes in when a player decides (in accordance with his opponent) to limit his designs, i.e. to behave in a certain way : logical formulas are therefore interpreted by *behaviors* —a behavior being a set of designs equal to its biorthogonal.
A last point : the text is by no means a philosophical discussion, not to speak of the scientific implementation of a philosophical viewpoint in the style of —say— Engels's *Dialektik der Natur...* It is has been a long time since philosophy has stopped interacting with logic, and our approximate use of a few philosophical concepts can only be metaphorical.

1 The old boulevard

Syntax is about language and semantics (from $\sigma\tilde{\eta}\mu\alpha$, sign) is about the interpretation of signs, an activity that includes

1. Divination from the bowels of animals, astrology, chiromancy, etc.
2. The job of translation : « *Traduttore traditore* ».
3. The search for meaning, which can be a mere illusion and is therefore likely to degenerate into item 1.

Technically speaking we are given on one hand a formal language (syntax) including formal *rules* (and axioms, which are rules without premises) enabling us to derive distinguished formulas, called *theorems* ; on the other hand, we are given a class of algebraic structures (semantics) including distinguished value(s), called *truth(s).* Syntax is interpreted in the semantics, with two desirable properties :

- *Soundness* : theorems are interpreted by truths
- *Completeness* : a formula whose interpretation is always a truth is a theorem.

The *ite missa est* of many papers reads like : A is provable in $\mathcal{L}$ iff A is true in any model of $\mathcal{L}$. The archetype is classical propositional logic (syntax) interpreted in the Boolean algebra $\mathbb{F}_2$, the only truth being the value 1. Should it be the only case, we would presumably not write this paper... the problem is that we have many [1] reasons to depart from classical logic, and that the alternative rules have always been justified by a completeness/soundness argument w.r.t. some class of structure. A real problem of *meaning*, in the sense of understanding what we are doing, mathematically, philosophically, sociologically etc. is posed : the implicit answer coming from the methodology of completeness/soundness is :

« *The meaning of logical rules is to be found in preservation of truth.* »

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[1] Too many indeed : only some of these reasons are good, but we don't know which ones.

The opposition proof/truth therefore organizes logic along one of those boulevards (form vs. contents, observation vs. world, soul vs. body, ideas [2] vs. matter etc.) so typical of Western thought up to the beginning of the $\mathrm{XX}^{\mathrm{th}}$ century.
The development of physics in the $\mathrm{XX}^{\mathrm{th}}$ century indeed took another direction :

1. Without contradicting in principle the deterministic paradigm *à la* Laplace, the theory of chaos denies any actual possibility of predicting —say— the output of the *roulette*.
2. The theory of relativity refuses the objectivity of some fundamental intuitions, such as time : time is now the quantification of causality w.r.t. some observer.
3. Quantum physics expels us from a paradise in which we can separate the observer from the phenomenon... without replacing it by a clear alternative.

The situation in logic is quite different ; the old boulevard is still there, although it lost part of its pregnancy :

1. Without questioning the objectivity of mathematical truth, incompleteness denies the actual possibility of mechanically solving —say— a well-chosen diophantine equation.
2. Intuitionism places the *subject* [3] at the center of mathematical activity : a proof of an existential statement $\exists n \phi[n]$ should induce an algorithm enabling *us* to extract an actual witness n.
3. The experience of (what is usually called) game semantics and/or linear logic suggests an even more radical departure from the familiar paradigm... which is the very point of this paper.

Our working hypothesis is that the emergence of new logical artefacts must induce a change of viewpoint [4]. I don't claim to achieve this here and if some of my propositions contradict one another, it might be ascribed to the peculiar period we are living : we have enough knowledge to depart from the old world but we don't yet know enough to produce a well-organized alternative... or maybe the situation is similar to that of quantum physics, i.e. the new world, although the right one, is definitely less comfortable than its predecessor.
Since we started this introduction with the alleged *motto* of the tradition, let us propose a *motto* for the alleged new wave. Our claim is that the opposition syntax/semantics has lost its pregnancy : in particular we shall carefully distinguish between several uses of the expression « semantics », which are far to correspond to the search for meaning : we shall conclude that, as we proceed towards the intimate meaning of logical operations, we are led to give up traditional semantics in favor of something which is closer to syntax. What we sum up in the provocative statement :

« *The meaning of logical rules is to be found in the rules themselves.* »

The problem will be to justify this *motto*, more precisely to justify it on non-trivial grounds, since this can be interpreted as a *gesticulation*, see below. Here we should perhaps think of traditional syllogistic which was essentially concerned with itself : the explanation of the figures (e.g. BARBARA, DISAMIS etc.) by their inner interactions is indeed the best explanation we can give of the meaning of these rules [5].

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[2] Even energy vs. matter, see e.g. Lenin's *Materialism and Empiriocriticism*, 1908.

[3] Brouwer used the expression « *Creative Subject* ».

[4] To be also read as *ideology* : the interest for consistency proofs, predicative correctness, etc., is a purely ideological expression of the old paradigm ; a new paradigm needs a new ideology.

[5] See the work of Abrusci, [2], relating syllogisms and non-commutative proof-nets, to convince one that there was something in the tradition.

2 The semantics of « semantics »

Behind the expression *semantics*, we distinguished divination, translation and perhaps meaning, which have their technical analogues :

2.1 Gesticulation

This is what happens when one produce a semantics just for the sake of producing a semantics. For a long time this was a trade mark of so-called philosophical logic : somebody produces a new operation, say $\diamondsuit$ (read as *Broccoli*) with some axiom, carefully chosen to be original, i.e. of no interest, say $(A \diamondsuit B) \Rightarrow ((A \diamondsuit A) \diamondsuit B)$, the *Broccoli* axiom, and then proves completeness and soundness w.r.t. all structures equipped with an operation $\heartsuit$ enjoying $a \heartsuit b \leqslant (a \heartsuit a) \heartsuit b$ [6]. Some parts of theoretical computer science have produced the absolute masterpieces of the trade, which —after all— have a paradoxical value : we can conclude from the semantics that the syntax under modelization is bullshit [7].
Gesticulation is surely not the major trend of semantics, but it has a certain importance, since aborted attempts at the search for meaning may end into gesticulation. In this respect, it will be our main enemy —prior to Tarskism, which is nothing but educated conservatism.

2.2 Treason

By this we mean the interpretation of logic in a model, preferably not the intended one : these devious interpretations are the very salt of model theory, and there is nothing to object to this noble activity... Semantics becomes a successful tool ; we can only object that this attitude is explicitly opposite to the search of meaning.

2.3 Tarskism

So when does semantics give meaning to rules ? Frege warned us —long time ago— that one must distinguish *Sinn* (meaning) from *Bedeutung* (denotation, i.e. standard semantics) : if $t = u$, then the two terms have the same interpretation, but we state this equality because t and u have different meanings, nobody being interested in $t = t$.
However there is a popular prejudice saying that there is something in Tarski's notion of truth : meaning might be located in the standard (i.e. not a devious one as above) interpretation, and we are back to the starting *motto*. Let us look at it : $A \wedge B$ is true when A is true *and* B is true, $\neg A$ is true when A is *not* true... and in the same spirit $A \diamondsuit B$ is true when A is true *Broccoli* B is true if *Broccoli* happens to make sense to us. Truth therefore commutes to all operations, i.e. the truth of A is expressed by A. To understand what is wrong here, let's have a look at a neighboring area : there might still be linguists which explain the French sentence « *Guillaume est étudiant* » by means of somebody called Bill, which happens to be a student... what a brilliant idea [8].
In fact the notion of *truth à la Tarski* [9] avoids complete triviality by the use of the magical ex-

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[6] Hint : interpret $A, B, \diamondsuit, \Rightarrow$ by $a, b, \heartsuit, \leqslant$.

[7] This is not a gratuitous joke : certain « logics » (non-monotonic, fuzzy, leap year, etc.) cannot be submitted to this minimal test for want of a decent syntax, i.e. of any deductive system... like a car without engine cannot fail a crash test. *Paralogicians* solves the problem of adequation between syntax and semantics by calling the semantics « syntax », whereas our gesticulators stay within reasonable bounds when they call the syntax « semantics ».

[8] Treason is more interesting, e.g. interpret « étudiant » by « Taliban ».

[9] Foreshadowed by the notion of « *vérité de La Palisse* », whose prototype is « *Un quart d'heure avant sa mort il était encore en vie* ».

pression « meta » : we presuppose the existence of a meta-world, in which logical operations already make sense ; the world of discourse can therefore be interpreted in the meta-world, typically the truth of A becomes « meta-A », and we can in turn explain « meta-A » by « meta-meta-A »... This is close to popular parapsychological explanations (e.g. in the movie « 2001 », where monkeys —us— have been educated by a galactical civilization) : the apparition of intelligence is explained by means of a meta-intelligence... QED. We are facing a *transcendental* explanation of logic « *The rules of logic have been given to us by Tarski, which in turn got them from Mr. Metatarski* », something like « *Physical particles act in this way because they must obey the laws of physics* ».
The abuse of the expression « meta » has completely distorted the relation of logicians to their own field... What to think of (educated) logicians who speak of « truth in the standard model » instead of plain truth (analyse the expression « standard ») or « intuitive integers » to mean plain integers... not to speak of this habit of concluding a completely trivial construction by « *But you know, it is* meta », the universally accepted excuse for the want of idea. We do not want to question the technical value of the distinction expressed by « meta », which is useful... but only its depth [10] : if everything useful were important, then we would spend our life speaking of *soap* and other hygienic artefacts.

3 Syntax

3.1 Against realism

If the truth analysis of A hardly says anything beyond A, this suggests that the truth of A does not make sense, being nothing more than a paraphrases of A. On the other hand, the syntactical properties of A —in particular the conditions under which A may or may not be derived— form a highly non-trivial interpretation for A. In fact everything is there, including classical models : a model of $\neg A$ is nothing but a branch in a *fair* proof-search tree for A. It is therefore legitimate, both technically and « philosophically » to say that everything is located in syntax, i.e. that the ultimate interpretation is syntactical : after all, whatever we do in mathematics, whatever are our fantasies when finding our way, we end with the production of signs that can —in principle— be reduced to pure syntax.

3.2 Consistency

The first person to realize this was Hilbert : his *formalist* explanation of logic only cares for *consistency*, i.e. the fact that the system will never collapse. This is a worthwhile effort in the direction of an *immanent* approach to logic : logical rules must be understood in terms of their inner harmony.
The traditional opposition between transcendence and immanence reads as follows in logic :

1. The fact that we must go outside a system to speak of this system (common sense [11] backed by Gödel's incompleteness theorem, 1931) is the irreducible transcendental part, the notorious « meta », a sort of Pascal's bet.

2. The fact that the laws of logic cannot be arbitrarily imposed from outside, i.e. that God himself cannot tamper with logic : one should develop criterions enabling us to judge logic from its internal structure.

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[10] See annex B.3 to see how the notion of truth etc. can be technically exploited.

[11] Something like « A hyperopic cannot fix his own glasses ».

The situation is indeed extremely intricate. In spite of its spectacular appeal, the limitation expressed by Gödel's theorem is not that deep, since we cannot anyway imagine a justification *ex nihilo* [12]. The real problem is with the criterions that we use to test our principles, for instance who told us that consistency is important, who is the judge of *internal harmony* ? Either you say « God » or « *It shows !* », and I do think that the second explanation is the only possible one, in spite of its obvious limitations.

There is yet another problem, not to be confused with the limitations coming from incompleteness. We basically propose an interactive [13] vision of logic. As we shall see, the notion of interaction between a strategy for A and a strategy for its negation seems more basic than the truth of A... but how can we explain the fact that we eventually seek *theorems* about games, whose main property is to be *true* ? *Basta così*, that's enough with pure methodology : any further discussion would either be technically circular, or enter into a classical philosophical debate —and we prefer not to argue with Mr. Hegel.

3.3 The great fear

The failure of Hilbert's program (Gödel's incompleteness theorem, 1931) expresses the inherent difficulty of his *formalism* : the consistency of $\mathcal{L}$ requires « more » than $\mathcal{L}$. My position on this kind of issue has just been explained : let us concentrate on the internal structure of logic and ignore the aspect « meta », i.e. that we *must* go outside $\mathcal{L}$ to speak about $\mathcal{L}$: for 70 years people have been obsessed by this question, and nothing came out of it... By the way, science also progresses by deciding that questions on which no progress is made are of no interest [14].

Let me develop this point : the obsession of (impossible) foundations is typical of logic. To ascribe it to the very nature of the logical activity is incorrect : similar questions occur in physics, e.g. « *What happened before the origin* », with a simple alternative, either invoke some *ersatz* of God, or ignore the question... and physicists obviously favor the latter solution, since astrophysics does progress. The real answer might be psychological, people with foundational anguish choosing rather logic than astrophysics : certain obsolete predications « *We protect you against lurking contradictions* » still find enough zelators ; on the other hand we live on a sort of wandering body with a ball of fire in its center, but nobody would buy an insurance against the explosion of Earth.

3.4 Algebraic completeness

The internal value of consistency is almost empty, think of all those consistent axioms which define (by completeness) algebraic structures that, hopefully, nobody will ever use : internal harmony of an axiomatic cannot be reduced to the existence of a model [15]. If we now remember that consistency comes « from outside », Hilbert's idea turns into a plea for transcendence.

... Unless we replace consistency with something richer. The obvious candidate for an inner interpretation is to interpret A by the set $A^{\bullet}$ of those hypotheses B which entail

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[12] Except in poetry, see *Igitur* where Mallarmé imagined a self-creation from emptiness ending into an ultimate throw of the dice, abolishing chance : « *Vous mathématiciens expirâtes* ».

[13] The right word should be *dialectic*, but it has been so heavily prostituted that we simply avoid it !

[14] Most people —including smart guys, remember Kepler— believe in the influence of stars, but science gave up for want of any tangible progress.

[15] An example of harmony taken from algebra is *associativity* : even Lie algebras have their form of association, the Jacobi identity.

it [16]. The idea passes successfully the test of classical completeness [17], the sets $A^\bullet$ forming a Boolean algebra : if we quotient this algebra by an ultrafilter, we are back to the usual $\mathbb{F}_2$-interpretation, which is considered as technically more interesting [18], but if we do not quotient, completeness is obtained w.r.t. a single well-chosen model. But as it is, without the quotient, it also works for many other logical systems, yielding Kripke models or topological models for intuitionistic logic, phase semantics [19] for linear logic... Are we approaching the evading meaning of logical operations, or are we developing a new kind of gesticulation ? The answer is so and so...

Indeed if I take logic $\mathcal{L}$, I can usually define a notion of $\mathcal{L}$-model : then the set of all $A^\bullet$ can be equipped with a structure of $\mathcal{L}$-model, the *free* $\mathcal{L}$-model, « free » being one way to state completeness w.r.t. this peculiar structure. This applies to the logic of *Broccoli*, based on the *Broccoli* axiom, and we can therefore construct the free *Broccolo* with its tautological completeness, a shame !

Wait a minute ! Who decided that completeness is so important ? It might again be one of those misconceptions [20] of (educated) logicians, and by the way it is. The theorem is seldom used, i.e. it hardly happens that we are given a theory, that we prove its consistency and then conclude that we have a model... It goes the other way around, we have the model, and it validates our syntax, i.e. we use soundness. Does this mean that completeness is useless ? Not quite, since it ensures that, in order to perform our favorite activity —treason—, we can stay inside $\mathcal{L}$-models. The fact that the peculiar $\mathcal{L}$-model used in completeness is an *ad hoc* free structure becomes less dramatic, since it is not the guy that we shall start with. From this remark we once for all (esp. in the program to be developed in section 5) seek trivial completeness results, non-trivial completeness being bad taste ; but we are also faced with the paradox that it might be highly non-trivial to discover which kind of structure might *naturally* (see below) harbor the trivial free $\mathcal{L}$-model [21].

Indeed soundness is difficult to satisfy, at least on non-contrived grounds. To understand this point, take phase semantics, see annex C : the data are a commutative monoid M together with an arbitrary set $\perp \subset M$, and completeness is achieved by means of the monoid M of *contexts*, $\perp$ being the set of those contexts which are provable in linear logic ; this is an unbelievably complex set, see [24], already NP-complete in the multiplicative case (result of Kanovitch). But an arbitrary pair $(M, \perp)$ validates the laws of linear logic, and the situation is as follows : a simple and natural class in which we can find a completely *ad hoc* universal element [22]. If we are now unhappy with linear logic and want to add some principle, then the same pattern will still hold, with the essential difference that we shall be forced to require that this additional principle holds in our semantical spaces... and we can see in annex C.4 that additional principles may behave well or behave bad, witness the fate of the unfortunate *Broccoli* axiom : in other terms the algebraic semantics, which interprets logic in terms of its provability is not that lax. Here, a simple criterion —whether or not a property of atomic *facts* $x^{\perp\perp}$ transfers to arbitrary facts— separates the wheat from the tares.

To come back to linear logic, the surprising thing is that its logical rules can be justified from

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[16] In sequent calculus, see Annex B, $A^\bullet$ is the set of all contexts Γ such that $\vdash \Gamma, A$ is provable.

[17] Whatever might be its limitations, classical logic remains The Logic, just as the catholic church remains The Church.

[18] Although the point in Paul Cohen's *forcing* is precisely to « delay » this quotienting.

[19] See annex C for basic definitions and properties.

[20] When using this word, it is impossible not to think of Kreisel's « current misconceptions ».

[21] Remember the Pierre Ménard of Borges, who becomes the author of the *Quichotte* as the result of a non-trivial reconstruction.

[22] By the way this is a genuine example of what our grandfathers —e.g. Poincaré— used to call *impredicativity*, which consists in defining something in terms of a set to which it belongs : provability defines a complex structure belonging to a set with a simple definition.

a very natural structure, a commutative monoid. We could however argue that somewhere we enforced commutativity and associativity by our choice… in fact nobody would seriously question associativity, but commutativity can and must be questioned. And again, our « immanentism » is comforted by what happens : if we take a general (non-commutative) M and an arbitrary $\perp$, then we get a mess, in which we eventually lose… associativity of the logical conjunction, see F.1. Now, if we require cyclicity, i.e. $xy \in \perp \Rightarrow yx \in \perp$, we get a very natural system, expounded as *cyclic linear logic* by Yetter [28]. The fact that cyclicity is natural should be obvious to persons with a basic mathematical culture, think of $Tr(uv) = Tr(vu)$.
Concretely the point behind the « naturality » of soundness is that one should be able to produce models in a fluent way : this is one of the key differences with gesticulation. And this is why phase semantics has often been useful to prove complexity results about logic, since it is easy to convert an abstract machine into a phase model, see e.g. the work of Yves Lafont, [22].

4 The monist duality

4.1 Proofs and tests

The immanent justice of Annex C.4 that fired *Broccoli* is important, i.e. we are now sure that if Mr. Metatarski had been crazy and had accepted —say the *Broccoli* axiom— as a « fundamental intuition » of the (meta-) universe, we would have detected the *bug*, in spite of preservation of truth. This means that our *motto* can be given a non-trivial sense, but we hardly got more.
Now, when we examine the (trivial) completeness argument, we see that what plays the essential part is *orthogonality* :

$$\Gamma \perp \Delta \quad \Leftrightarrow \quad \vdash \Gamma, \Delta \ \text{ is provable}$$

Formulas interact through provability, and this gives rise to a complete interpretation. However, the algorithmic contents of cut-elimination favors the notion of proof, for instance, Booleans are represented by means of $A \vee B$ (or $A \oplus B$), with exactly two cut-free proofs, see Annex B : but although orthogonality based on provability can distinguish this formula from many other ones, it will never *separate* its two proofs. We need a (separating) duality involving proofs, see Annex A.3, and we shall start with a duality between proofs and *tests*.
We now explain the notion of proof by means of the notion of test : for this we revisit the *semantics of proofs* developed around 1930 by Heyting and Kolmogoroff :

1. According to Heyting, a proof π of the conjunction $A \wedge B$ consists in the data of a proof π_1 of A and a proof π_2 of B. If I want to test $\pi = (\pi_1, \pi_2)$, a would-be proof of $A \wedge B$, I need to test both components : if π passes all tests, those on A as well as those on B, then π is a proof of the conjunction. We conclude that a test for $A \wedge B$ is a test for A *or* a test for B. Observe the ESSENTIAL point that the notion of test is subtler than the classical notion of refutation (counter-model) : a countermodel refutes A or B, i.e. may refute both, whereas the test attacks A (left) or B (right), but not both of them… In particular a test for $A \wedge A$ is a pair (τ, i), where τ is a test for A to be applied against π_1 if $i = 1$, against π_2 if $i = 2$.

2. This must be related to Heyting's semantics of proofs of the disjunction : a proof of $A \vee B$ is a pair (π, i) with either $i = 1$ and π is a proof of A or $i = 2$ and π is a proof of B. If

I want to test a would-be proof of $A \vee B$, I have to prepare two tests, one in case (π, i) pretends to be a proof of A $(i = 1)$ one in case it pretends to be a proof of B $(i = 2)$: a test for $A \vee B$ is therefore a pair (τ_1, τ_2) of a test for A and a test for B : this looks like the semantics of proofs of a conjunction.

3. Heyting's semantics of proofs of implication says that a proof π of $A \Rightarrow B$ is a function from proofs of A to proofs of B ; a test for such a proof therefore consists in the data of a proof π' of A (the argument of function π) and a test for B, to check that $\pi(\pi')$ is a proof of B. We see that testing an implication is like proving a conjunction.

We can summarize our remarks by means of the (temporary) symbol A^t for tests on A :

$$(A \wedge B)^t = A^t \vee B^t \qquad (A \vee B)^t = A^t \wedge B^t \qquad (A \Rightarrow B)^t = A \wedge B^t$$

and these formulas are reminiscent of the familiar De Morgan laws of classical logic :

$$\neg(A \wedge B) = \neg A \vee \neg B \qquad \neg(A \vee B) = \neg A \wedge \neg B \qquad \neg(A \Rightarrow B) = A \wedge \neg B$$

This analogy suggest an identification between A^t and $\neg A$, together with a duality based on the analogy :

$$\textit{test for } A \quad \sim \quad \textit{proof of } \neg A$$

If this works, then a test will appear as a generalized proof, a sort of « paraproof ». But this means that we are now establishing a duality between homogeneous objects, what we call a *monist duality*.

4.2 The symmetrization of intuitionistic logic

This approach fails in the intuitionistic case, for which is was tailored, since negation is too badly behaved : for instance this connective is not *involutive* (i.e. $\neg\neg A$ is not equivalent to A). This is not because intuitionists defined the wrong negation, this the result of a geometrical limitation : the only meaning of negation is the symmetry around the symbol $\vdash$ of sequent calculus, and its involutivity would mean that the left and right zones are isomorphic... but on the right-hand side only one formula is allowed, whereas on the left hand side we can use an arbitrary number of formulas, and we can therefore use the « structural rules » of *contraction* and *weakening* :

$$\frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A} \qquad\qquad \frac{\Gamma \vdash A}{\Gamma, B \vdash A}$$

The rules for negation enable a formula A to transit from right to left and « benefit » from structural rules under the disguise of $\neg A$, before coming back to right : but it is now $\neg\neg A$, the double negation betraying its excursion to the left. Concretely double negation enables one to combine several attempts at proving A, typically two attempts at the excluded middle $B \vee \neg B$ to prove the double negation $\neg\neg(B \vee \neg B)$. By the way, classical logic which symmetrically admits several formulas to the right of $\vdash$ (as well as the corresponding structural rules) differs from intuitionistic logic in the dispense of the transit to left, and therefore of the writing of the prefix $\neg\neg$; its non-constructivity, i.e. the failure of the *disjunction property*, see Annex B, comes from the fact that a disjunction $A \vee B$ is obtained as a mixture of attempts to get A and to get B, an *imbroglio* impossible to unwind in general.

Our identifications suggest a symmetrization of intuitionistic logic ; one formula to the left would be too drastic, hence we must, like in classical logic, admit several formulas to the right, i.e. sequents $\Gamma \vdash \Delta$. The danger is to rediscover boiling water, i.e. classical logic which admits no (standard) constructive interpretation, in particular no separating duality. But we can also consider that the restriction « one formula to the right » is nothing but a hypocritical way to forbid right structural rules... hence we can also restore symmetry by forbidding structural rules on both sides. The resulting symmetrized intuitionistic logic is *linear logic* [23].

Linear logic is indeed an enforcement of the paradigm of semantics of proofs : a proof of an implication is now a *linear* function, in a sense close to linear algebra, see Annex D. This shift of viewpoint is a strong restriction on implication, now called *linear implication*, with a new symbol $A \multimap B$. Linear negation, noted $A^\perp$, is obtained by analogy with the dual space in linear algebra, and is involutive. In analogy with linear algebra which admits two products, linear logic admits two conjunctions, & (*with*, a direct sum, which corresponds to intuitionistic conjunction) and $\otimes$ (*times*, a tensor product). Involutive negation enables one to access, through duality, to two disjunctions :

$$A \oplus B = (A^\perp \,\&\, B^\perp)^\perp \qquad\qquad A \parr B = (A^\perp \otimes B^\perp)^\perp$$

respectively called *plus* [24] et *par*. The principles $A \multimap A \otimes A$ and $A \otimes B \multimap A$, which correspond to the two structural rules are interpreted by non-linear functions (the former is quadratic, the latter is affine) and are therefore wrong ; the connective « ! » (*of course*), built in analogy with the symmetric tensor algebra, is introduced to compensate for the want of structural rules. Intuitionistic implication becomes a defined connective :

$$A \Rightarrow B = (!A) \multimap B$$

the original equation from which linear logic was extracted.

But let us come back to tests : linear negation $A^\perp$ must correspond to tests on A. A test for $A \,\&\, B$ is a test for A or a test for B, i.e. $(A \,\&\, B)^\perp = A^\perp \oplus B^\perp$, whereas a test for $A \otimes B$ is a linear function from A to $B^\perp$, i.e. $(A \otimes B)^\perp = A^\perp \parr B^\perp = A \multimap B^\perp$, so that the identity function from A to A is a test for $A \otimes A^\perp$ (see footnote 25 for its game-theoretic interpretation). The two conjunctions are tested in a different way : in the first case only one component is tested, whereas in the second case the two components are put into « contact » by the test. There are therefore two utilizations of conjunction (and dually of disjunction), and this diversity is legalized by the splitting of connectives. In particular there is a big difference between the two « $\wedge$ » in

$$(A \vee B)^t = A^t \wedge B^t \qquad\qquad (A \Rightarrow B)^t = A \wedge B^t$$

Moreover, in the first case, the test for A^t is here for at most a single use, whereas in the second case, the proof of A can be used as many times as needed : linear logic would write $!A$ in this case, i.e. $(A \Rightarrow B)^\perp = {!A} \otimes B$. These subtle points were overlooked in our discussion of the intuitionistic case ; in fact the impossibility of staying inside intuitionism can be attributed to the need to be careful about multiple uses of tests : this original feature of linear logic is known as *resource-sensitivity*.

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[23] The symmetric character of linear and classical logics makes it possible to put everything to the right : this is a significant simplification of the calculi, and it is the choice we made in Annex B.

[24] Linear logic can also be modelized in Banach spaces, see [17] ; the difference between & and $\oplus$ is expressed by distinct norms, ℓ^∞ and ℓ^1. Exponentials are obtained through analytic functions on the unit ball, in particular $!X$ is the space of kernels of the sort *Cauchy integral*.

4.3 Games

But still remains the problem of the correctness of the monist duality between A et $A^{\perp}$: it cannot be phrased *proofs of A/proofs of* $A^{\perp}$ in the literal sense, since it is impossible that both a formula and its negation are provable : since *duality abhors a vacuum*, we arrive at the conclusion that tests cannot be reduced to plain logical proofs. To stress the essential unity (monism) of the interpretation, we now change our expression « test » into *paraproof*, i.e. a test for A is a *paraproof* of $A^{\perp}$.
What could be the status of those lurking paraproofs ? Here *game theory* provides us with a major intuition (what follows is not a precise technical definition, since games may differ on many technical details, and we do not want to commit ourselves).
Two players (I sometimes called *Player* and II « *Opponent* ») compete, and we shall adopt the viewpoint of I. We can assume that both players follow *strategies* and if I and II play according to respective strategies σ and τ, the result is a *play* $\sigma\tau$. A play has at most one winner, I or II. Among the strategies σ for I we can look for *winning* ones, i.e. those for which $\sigma\tau$ is won by I whatever τ is played by II. This yields the following dictionary :

$$\begin{array}{lcl} Rule\ of\ game & = & Formula \\ Strategy & = & Paraproof \\ Winning\ strategy & = & Proof \\ \sigma\tau & = & Result\ of\ test\ \tau\ applied\ to\ \sigma \end{array}$$

With this interpretation, logical connectives correspond to forms of socialization of games, i.e. possible ways to combine them and produce new games : typically linear negation is the interchange between players I and II. If A and B are games, then $A \,\&\, B$ is played as follows : first II chooses one of the two games A or B, then the play proceeds according to this choice as a play in A or a play in B ; we see that a strategy for II is a strategy for A or a strategy for B, whereas a strategy for I is a pair of strategies etc. In the cotensor $A \parr B$, II will combine a strategy for A and a strategy for B, since I will be allowed to switch from one game to the other : typically, in the case of $A^{\perp} \parr A$ (i.e. $A \multimap A$), the *Opponent* II uses both a strategy σ for A and a strategy τ for $A^{\perp}$, (which is a *Player*'s strategy for A) ; then I has an excellent strategy, which consists in playing σ against τ, by a simple import/export of moves... in this way the principle $A \multimap A$ is justified [25] : this *copycat* strategy based on imitation is indeed an implementation of the identity map.
The idea is therefore very appealing ; but, as soon as we want to formalize it, we are forced to enter into endless questions : temporality of the board —who plays first, do players alternate, are plays finite or infinite ?—, the nature of the board —numbers, space ?—, the nature of gain —possibility of a draw, numerical gain i.e. money ?—, the nature of the strategies —e.g. memory-free—, to see that the question is not that simple. All extant notions, e.g. the games of Blass [5] validate the laws of linear logic, but also other laws that are sometimes completely unacceptable, e.g. classically wrong.

4.4 The school of Lorenzen

It is only now that one starts to realize that Lorenzen [26] had —around 1960— his own prefiguration of this program. His school remained indeed very marginal, mainly illustrated by his student Lorenz [25], with some tardive work by Felscher [8] in the mid 80's ; contrarily to intuitionism which managed to keep a live tradition and to have some influence in

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[25] This amounts to play against Kasparov (with the Whites) and Karpov (with the Blacks : negation exchanges black and white) : when Kasparov plays, I recopy his move in the play against Karpov and *vice versa*. I cannot lose both plays ! With a reasonable notion of gain, this is indeed a winning strategy. Moreover, it is a typical logical approach, i.e. it is independent from the actual rules of Chess.

the revival of intuitionistic ideas, this school completely disappeared, so we don't have any feeling from the inside.

It seems that the program was connected to the idea of a *dialectic* interpretation of intuitionistic logic : cut-free proofs are seen as *dialogues* between a proponent which produces right logical rules (answers) and an opponent that produces left logical rules (questions). As we just explained, the symmetry between players is incompatible with intuitionistic logic, hence the position was not quite comfortable ; one can also guess that Lorenzen might have been doubly marginal, as doing intuitionism in Germany and as taking a dialectic approach to intuitionism. Moreover the technical achievements remain modest : for instance Felscher had not the slightest sense of *compositionality* of strategies, which should be central in such an enterprise (it is the semantic content of cut-elimination). By the way Hugo Herbelin has tried to understand Felscher's work in non-bureaucratical terms, and individualized the use of specific moves for ternary operations like $A \wedge B \Rightarrow C$ and $(A \Rightarrow B) \Rightarrow C$, but not for their 4-ary analogues : this technicality induces completely different treatments of —say— $(A \wedge B) \wedge C \Rightarrow D$ and $A \wedge (B \wedge C) \Rightarrow D$, i.e. produces a non-associative conjunction !

Although the technical record of the school seems rather bleak, it would be unfair not to acknowledge the fact that Lorenzen had understood part of the picture before us ; other parts of the same picture were hidden in the Curry-Howard isomorphism, in denotational semantics… but his troup vanished before any junction could be established.

5 Methodological commitments

Our goal is therefore to develop an interpretation in the spirit of game semantics and to prove the adequation of this interpretation w.r.t. linear logic. Technically speaking this adequation takes, as usual, the form of a dyptic completeness/soundness, although the spirit is very different. The idea is basically to abstract enough from syntax (getting rid of irrelevant bureaucracy, e.g. brackets, see the discussion in [15]) so as to get a geometrical object which is a game. We should arrive at some place in-between where the distinction syntax/semantics vanishes.

For what follows, we assume that formulas A, proofs π, are interpreted by means of games $|A|$ and strategies $|\pi|$.

5.1 Full completeness

Completeness w.r.t. games should be stated as follows : *if σ is a winning strategy for the game $|A|$, then there is a proof π of A such that $\sigma = |\pi|$.*

By the way Gentzen is not only responsible for sequent calculus, but also for the game interpretation of logic… In his immature first consistency proof, [10, 11] of arithmetic (1936), he interprets proofs by strategies in a game : instead of just saying that A is true, the strategy gives an interactive way to check its truth against any opponent. It is irrelevant to notice that in reductionist terms this « proof » (strictly) contains the flat justification of the rules by their truth… one should only remember that Gentzen managed to give a non-tautological —i.e. non-Tarskian— meaning to formulas : for instance there can be several strategies for the same formula, whereas there is only one truth value. However, his approach was far astray from a systematic and well-understood explanation of logic in terms of games, and we have seen what happened to the school of Lorenzen.

Now if we want to be serious, the first thing is to see whether this completeness statement is reasonable. I claim that completeness should be restricted to the case where A is a closed

Π^1 formula, see Annex A.1. I give two reasons that seem to be beyond discussion :

1. It must be technically possible to define the *truth* of A as the *existence* of a winning strategy. Then full completeness implies ordinary completeness, hence A must be Π^1.
2. Take for instance a closed Σ^1 formula A. Its proofs depend on the (higher-order) logical system under consider ; since each proof induces a winning strategy and none of these higher-order systems can exhaust all possibilities, completeness fails. The situation is different when A is Π^1, since, by the *subformula property*, see Annex B, all the proofs are « already there », an internal completeness.

Indeed, the subformula property only applies to cut-free proofs. Hence we end with the following reformulation of completeness :

If A is a closed Π^1 formula and σ is a winning strategy for the game $|A|$, then there is a cut-free proof π of A such that $\sigma = |\pi|$.

5.2 Paraproofs and paralogisms

As we said, completeness *must* be trivial, the non-trivial efforts being concentrated on soundness. This basically means that we shall try to find a very simple, natural, geometrical... notion of game, and that we shall shamelessly disguise proofs as such games and get completeness without efforts. Proofs will yield winning strategies, but what about general strategies ? They must also come from syntax. This leads to the following extraordinary conclusion : paraproofs must be witnessed in the syntax, and there is only one way to do so, namely to admit MISTAKES of logic. In other terms, paraproofs are just proofs in a larger system, in which some *paralogisms* are admitted. But not any crazy principle will access the dubious honor to be admitted as a paralogism : certain crucial properties must be satisfied, namely :

1. Cut-elimination still holds.
2. Paralogisms produce « enough » paraproofs.

The first commitment comes from the fact that cut-elimination will play a prominent role, so we should look for principles that preserve cut-elimination. The second condition explains the absolute need to have some real paraproofs : after all our game interpretation will represent the interplay between I, who tries to prove A and II, who tries to prove $A^\perp$; without paralogisms, this cannot work, since at least one of the two guys cannot even pretend (A and not $A^\perp$ cannot be both provable, and in general none of them is provable). One typical paralogism that respects cut-elimination is the axiom $\vdash \Gamma$, for any sequent Γ (in terms of games, it is the move « I give up ») and it is definitely incorrect. However, this move is not that strange, if we think of proof-search : starting from the conclusion, we try to prove $\vdash A$, and at some moment, we might give up, and why shouldn't we give a status to this aborted proof ? But this might not be enough, as shown by the following example : I tries to prove $\vdash ?A \otimes ?B$, hence II tries to prove $\vdash !A^\perp \parr !B^\perp$, which amounts to proving $\vdash !A^\perp, !B^\perp \ldots$ but there is no rule with this conclusion, but the paralogism « Give up » ; if II can only give up, then I will win $\vdash ?A \otimes ?B$ without any effort. There are two possible paralogisms that II might use to respond, namely *weakening* (premise $\vdash !A^\perp$ or $\vdash !B^\perp$) or *mix* (premises $\vdash !A^\perp$ and $\vdash !B^\perp$). Both of them are compatible with cut-elimination, although their geometrical behavior is different : weakening induces an irreversible destruction of

information, whereas mix destroys the connexity of the underlying geometrical pattern. There is a third possibility, namely to replace linear logic with an *affine* version allowing weakening, in which case « Give up » would be the only paralogism [26]. Only the complete achievement of the program —i.e. a non-contrived soundness— can decide between these three alternatives, and by the way observe that our methodology, in case the third possibility makes it, would force us to conclude that linear logic is wrong, i.e. should be replaced with its affine variant... Unpleasant issue, but if we are not ready to accept such consequences, then we are not earnestly seeking the ultimate meaning of logic.
Our *motto* is fully justified by what we just explained. But it is slightly provocative, since we mean something more, i.e. that rules have a geometrical meaning : this will be the main concern in the search for soundness, which is highly non-trivial. Perhaps one should amend it into :

The meaning of logical is hidden *in the rules themselves.*

5.3 Soundness

Soundness reads simply as :

If π is a paraproof of A, then $|\pi|$ is a strategy for the game $|A|$; moreover if π is a proof, i.e. uses no paralogism, then $|\pi|$ is winning.

We must find a *natural* notion which *harbors* the more or less trivial game induced by syntax, the difficulty being to satisfy both requirements, natural notions having a tendency to shun the trivial game.

5.4 Geometry of interaction

Indeed this program originates in the analysis of the correctness criterion for *proof-nets* of [12], done in [13], see Annex E. The question was to handle a new kind of parallel syntax, but this seemingly syntactical question received a purely geometrical answer : at that very moment, I understood that the interpretation of the rules is hidden in syntax... well-hidden indeed.
Under some mild hypotheses, a cut-free proof of a purely multiplicative formula A (using literals $p, p^\perp, q, q^\perp, \ldots$ and the connectives $\otimes$, $⅋$), makes use of axioms linking each occurrence of a literal p with an occurrence of its negation $p^\perp$. This linkage, which is the only actual information contained in the proof, can be seen as a permutation σ of the literals of A, enjoying $\sigma^2 = 1$. On the other hand, a *switching* $\mathcal{S}$ of the proof-net induces another permutation $\tau_\mathcal{S}$ of the same proof-net, and the correctness condition can be rephrased as « $\sigma\tau_\mathcal{S}$ cyclic for all $\mathcal{S}$ ». It is possible to consider the $\tau_\mathcal{S}$ as paraproofs of $A^\perp$, it is also possible to distinguish *winning* ones etc. and the correctness theorem of [12] can be read as a full completeness result —together with a natural soundness— for multiplicative linear logic, the basic orthogonality being defined by :

$$\sigma \perp \tau \quad \Leftrightarrow \quad \sigma\tau \text{ is cyclic}$$

Annex E.7 presents this result in a simplified form, permutations being replaced with partitions.

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[26] Paralogisms other than « Give up » have the property of making your opponent lose, at the price of your own victory : this is rather mean, like the notorious gardener's dog that does not eat salads, but bites you if you try. I therefore propose to call them « Dog's moves ».

Generalizing this result was very tantalizing, and the problem was indeed posed —perhaps in cryptic terms— in [15], as the program of *Geometry of Interaction* (GoI, 1987). The main difficulty was to accommodate exponentials ; for this purpose, permutations were replaced with unitary operators on Hilbert space. This worked quite well, see e.g. [14], and had even some rather non-trivial applications in theoretical computer science ; in particular it is deeply related to the game interpretation of Abramsky & al. used to solve the question of *full abstraction*, see [1]. But definitely this is not the answer : σ, τ become operators, the orthogonality being defined as :

$$\sigma \perp \tau \quad \Leftrightarrow \quad \sigma\tau \text{ is nilpotent}$$

Indeed nilpotency is quite different from cyclicity, and there is no longer any way to separate the wheats from the tares, i.e. to define a reasonable notion of *winner*.
I eventually arrived (through other considerations, see below) at the conclusion that time was the great absent of GoI. One possible reconciliation is that the operators of GoI should be presented as inductive limits of finite dimensional ones, the process of inductive limit being precisely this neglected time.

5.5 Coherent semantics

There is an alternative approach, based on *coherent semantics*, see Annex D. Coherent semantics is the original interpretation of linear logic, in the sense that linear logic was extracted from it. The basic paradigm is that formula A becomes a coherent space $|A|$ (i.e. a graph) and that a proof π of A becomes a *clique* $|\pi| \sqsubset |A|$. Since $|A^\perp|$ is the complementary graph, proofs of $A^\perp$ become *anticliques* of $|A|$. The problem is that many cliques (typically the empty one) of $|A|$ cannot represent a proof : we must therefore find a notion of *winning* clique. This is why we can try to enhance our coherent spaces with *totality* (CST), see Annex D.5.
What follows is essential —at least to anybody interested in the problem— : I will open my toolbox and list the seminal counterexamples on which I worked for years. They are all located in the multiplicative/additive world, and are therefore extremely simple ; each of them presents a different facet of the question [27].

5.5.1 Tensor/False

The formula $A \otimes \perp$ is problematic. In fact it is enough to consider the formulas generated from $\mathbf{1}, \perp$ by means of $\otimes$, $\parr$ to stumble on a serious problem : all these spaces are isomorphic CST, but not if we also consider a notion of *gain*, see D.5. If we choose $0-1$-gain, then our atoms will obey to classical logic, if we choose a gain in $\mathbb{Z}$, then $\mathbf{1}, \perp, \otimes, \parr$ will receive the interpretations $1, 0, x+y-1, x+y$, an interpretation known to be incomplete (counterexample of Fleury, 1988 : $\perp \otimes (1 \parr 1)$). In fact the gain space should be endowed with a structure of phase space, a bad start indeed, if we want to avoid gesticulation.

5.5.2 Tensor/True

The consideration of the additive neutrals is a nightmare, because duality abhors a vacuum. The dual of $A \otimes \top$ is $A \multimap \mathbf{0}$, and as explained in the Annex, a total element in $A \multimap \mathbf{0}$ should map total elements of A to total elements of $\mathbf{0}$, which is impossible, unless A has no total element. But if $A \multimap \mathbf{0}$ has no total element, then any clique in $A \otimes \top$ (there is only one,

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[27] These problems are not creations of linear logic : they already exist in intuitionistic logic, typically the case Tensor/true below deals with the familiar degeneracy of intuitionistic negation.

the empty set) is total and winning. In other terms, as soon as A has some total element, i.e. some paraproof, $A \otimes \top$ should be provable… nonsense !

5.5.3 Plus/True

Consider now $A \oplus \top$; its dual is $A^\perp \,\&\, \mathbf{0}$, and from the definition of the totality in a « & », the dual has no total element. As a consequence, every clique in $A \oplus \top$ is total and winning. Completeness fails here too, although not that dramatically : we can identify two cliques which cannot be separated by the *Opponent*, and things start to get fixed… with a lingering problem : we can no longer distinguish between the two proofs of $\mathbf{1} \oplus \top$.

5.5.4 The Gustave function

This fantastic counterexample is due to G. Berry ∼ 1978. It is a contribution to the theory of sequentiality : the precise semantical definition of what is a *sequential* function is the algorithmic analogue of our concern for full completeness. The Gustave function takes three Booleans arguments and returns a completely irrelevant output. The equations are the following

$$\begin{array}{rcl} G(\mathrm{true}, \mathrm{false}, z) & = & a \\ G(x, \mathrm{true}, \mathrm{false}) & = & b \\ G(\mathrm{false}, y, \mathrm{true}) & = & c \\ G(\mathrm{true}, \mathrm{true}, \mathrm{true}) & = & d \\ G(\mathrm{false}, \mathrm{false}, \mathrm{false}) & = & e \end{array}$$

(the last two equations have been added to Gustave's definition to make the function total). The algorithm thus defined is not *sequential*, i.e. when we compute G, we have no first question to ask about the input (e.g. « *Give me the first argument* » : in case $y = \mathrm{true}$, $z = \mathrm{false}$, then the first argument is irrelevant). Of course, if one replaces the second equation with

$$\begin{array}{rcl} G(\mathrm{true}, \mathrm{true}, \mathrm{false}) & = & b \\ G(\mathrm{false}, \mathrm{true}, \mathrm{false}) & = & b \end{array}$$

then sequentiality is restored.

Let us transform these equations into a problem of linear logic : any function of boolean arguments can be made linear by changing its input space from Bool to $\mathbf{1} \,\&\, \mathrm{Bool}$, and remember that Bool can be taken as $\mathbf{1} \oplus \mathbf{1}$. We can therefore see the function as a linear function defined on the ternary tensor power of $\mathbf{1} \,\&\, (\mathbf{1} \oplus \mathbf{1})$. The output is irrelevant. The Gustave function is transformed into an equivalent counterexample, with a purely logical content : Let A, B, C be three CST, and consider the CST $D = (A \oplus (B \,\&\, C)) \parr (A \oplus (B \,\&\, C)) \parr (A \oplus (B \,\&\, C))$. As a coherent space, it contains an isomorphic copy of the « & » of five CST, namely, $B \parr C \parr A, A \parr B \parr C$, $C \parr A \parr B, B \parr B \parr B, C \parr C \parr C$. Now if I take a total clique in each of these CST, then their « union » is easily shown to be a total clique in D, moreover, in case my cliques are winning, their union is winning too. But this clique cannot correspond to any proof, since we cannot find a last rule ending with the sequent $\vdash A \oplus (B \,\&\, C), A \oplus (B \,\&\, C), A \oplus (B \,\&\, C)$ which would correspond to such a clique : indeed six $\oplus$-rules might apply, corresponding to the removal of one of the three A or one of the three $B \,\&\, C$, but our clique needs these six guys.

In fact, Berry studied another counterexample, the *parallel or* of Plotkin

$$\begin{array}{rcl} P(\mathrm{true}, y) & = & a \\ P(x, \mathrm{true}) & = & a \end{array}$$

which can be interpreted as a subset of $(A \oplus B) \parr (A \oplus B)$, the union of a clique in $A \parr B$ and a clique in $B \parr A$; but the union is not a clique, and this example is killed by coherent spaces[28]. The case of Gustave is much more delicate : Ehrhard found a beautiful generalization of coherent spaces, *hypercoherences*, [7], which manages to kill Gustave's function, but it seems that this is not the right way to full completeness.

5.6 Time

Monastic life helps to put things together ; after weeks of meditation (on this problem) I realized that the completeness argument would anyway be trivial. So what is the main difficulty ? Essentially to construct the proof π such that $\sigma = |\pi|$, and this essentially amounts at finding the last rule of π : if property $\mathcal{P}$ ensures the existence of a last rule and if the premises of the rule inherit property $\mathcal{P}$, then we can iterate, and we are left with a more tractable problem, namely to show that an existing would-be proof is a proof. The two seminal examples of Berry are indeed failures of the same principle, the last rule : the parallel or is the case of a sequent $\vdash A \oplus B, A' \oplus B'$, the Gustave function being that of a sequent $\vdash A \oplus B, A' \oplus B', A'' \oplus B''$ (renaming $B \,\&\, C$ as B, etc.). The property which fails is the possibility to simplify one of the « $\oplus$ » into one of its components : this corresponds to a logical rule for « $\oplus$ », six possibilities for *Gustave*, four possibilities for *parallel or*. By the way, the four possibilities of the parallel or are analogous to the Whitman property of lattices : if $a \wedge b \leqslant c \vee d$, then either $a \leqslant c \vee d$ or $b \leqslant c \vee d$ or $a \wedge b \leqslant c$ or $a \wedge b \leqslant d$. The Gustave function shows that the ternary analogue of the Whitman property : if $(a \wedge b).(c \wedge d) \leqslant e \vee f$, then $a.(c \wedge d) \leqslant e \vee f$ or $b.(c \wedge d) \leqslant e \vee f$ or $(a \wedge b).c \leqslant e \vee f$ or $(a \wedge b).d \leqslant e \vee f$ or $(a \wedge b).(c \wedge d) \leqslant e$ or $(a \wedge b).(c \wedge d) \leqslant f$ is independent from the binary case.

What is the general pattern ? The solution is linked to a splitting of the connectives in two halves, the *positive* ones, here $\mathbf{0}, \mathbf{1}, \oplus, \otimes$ and the *negative* ones, $\top, \bot, \&, ⅋$. This seems to be a graphical joke : indeed the graphism was chosen early in 1986 —with the benediction of Yves Lafont— so as to individualize two groups, one written in an algebraic style, the other in a logical style, the symbol $(.)^{\perp}$ for negation belonging to both of them. The main advantage of this notation is to mnemonize the canonical isomorphisms of Annex D, e.g. distributivity. But there is a more general explanation for all these isomorphisms : connectives of the same *polarity* do *associate*, see Annex F, in a sense which basically covers the isomorphisms of Annex D.

Association was first discovered by Andreoli (the only extant producer of software based on linear logic), under the name of *focalization*, [4], which explains (in terms of logic programming, i.e. proof-search) the algorithmic contents of the biorthogonal. In fact double orthogonals are prominent in most of our semantical artefacts, phase semantics, CST, game semantics etc. A definition with too many of them is unmanageable, hence one needs to reduce their number ; for instance, associativity of « $\otimes$ » comes from the possibility of simplifying $(A \otimes B^{\perp\perp})^{\perp\perp}$ into $(A \otimes B)^{\perp\perp}$, see Annex F.1. If we remove biorthogonals from our basic definitions, we see that our operations split according to two polarities, the *positive* ones satisfying $\Phi(X^{\perp\perp}) \subset \Phi(X)^{\perp\perp}$[29]. The only place where biorthogonals are actually needed is when we apply a negative operation after a positive one : *this is the basic scansion of time in logic.* A logical step therefore appears as a cluster of positive operations following a cluster of negative operations ; in fact we should rather count two steps here, namely *Opponent*'s move followed by *Player*'s move.

Looking back at the basic counterexamples of last subsection, one can see that they are indeed all due to an alternation of polarity, e.g. $\bot$ followed by $\otimes$, i.e. in all these cases time is guilty.

The pregnancy of time forced me to depart from the synthetic viewpoint of geometry of

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[28] Originally by Berry's notion of *stability*.

[29] In the Banach space interpretation, positive means ℓ^1-norm, negative means ℓ^∞-norm.

interaction, in particular Hilbert space operators. This regression is surely not definitive, after all GoI is the conceptualization of a previous combinatorial experience.

5.7 Hypersequentialization

As a consequence of Andreoli's focalization we can modify the rules so as to iteratively apply —either the maximum negative cluster of rules (this is a mechanical step corresponding to the time when the *Opponent* will move in the actual play) —or a maximal positive cluster of rules (which is by no ways mechanical, creativity is here at work : this corresponds to a *Player*'s move.) If we stick to this principle, we see that the positive and negative steps alternate and that sequents always contain at most one negative formula. Indeed something outstanding happens : cut-elimination can be performed without any permutation of rules, unbelievable ! . . . especially if one remembers that the plague of sequent calculus, the reason why Natural Deduction, Proof-nets etc. were introduced, is to minimize permutations of rules, see Annex E. Now we got our ultimate syntactical object : those *hypersequentialized* proofs.

5.8 Deconstruction is achieved

We achieved our « deconstruction »[30] of proofs. A (para-) proof is a proof in the hypersequentialized calculus, i.e. —starting from the conclusion— a succession of (clusters) of rules, maybe never ending, i.e. infinite. We now have to reconstruct everything from this basic artefact, i.e. to proceed with the synthesis. This will involve a translation of everything into pure combinatorics, in a way keeping syntax astray.

Our (para-) proofs will become strategies, and strategy σ will be seen as the set of all plays $\sigma\tau$, when τ varies through all strategies for II. So what is play ? If σ, τ stand for (para-) proofs of $\vdash A$ and $\vdash A^{\perp}$, then the play must have something to do with the paraproof of the empty sequent « $\vdash$ » obtained by cut. Indeed, if we start to eliminate cuts in this sequent, starting from the conclusion, we produce a sequence of operations which, due to the alternation of polarities in σ and τ, looks like the alternate moves of two players.

Here something must be made to avoid gesticulation : uniform decisions must be taken as to the games. For instance, we decide that all plays are finite, since the notion of gain is problematic in the infinite case (concretely this means that the rule « Give up » plays a prominent role). Also the notion of gain must be defined once for all : a player loses when he gives up or makes a *dog's move* (hence most of plays are indeed draws), and dog's moves must be defined in a geometrical way, e.g. as the destruction of connectedness.

How do we recognize the would-be strategies ? Simply by building a universal (sort of) coherent space : two plays are coherent (from the viewpoint of I) when II is responsible for the fact that they differ : coherence will therefore enforce the existence of a uniform first move. Orthogonality will be defined as $\sigma \cap \tau \neq \emptyset$. By the way, these strategies are not general strategies, and a new word should be found : I propose « design », a design for I and a design for II inducing a *dispute*.

In fact we have produced a single universal game, so we are not quite explaining formulas by games. If I and II use the full range of possibilities nothing can work : orthogonality cannot be respected (technically speaking, $\sigma \cap \tau = \emptyset$, which actually means that the two designs have produced an infinite dispute). We define a *behavior* as a set of designs equal to its biorthogonal, i.e. a behavior for I is defined as the orthogonal of a behavior for II and vice-versa. This means that the two players have somewhere agreed not to use the

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[30] Less emphatically : our analysis.

atomic weapon. What does this actually mean, besides the mathematical definition ? The only analogy I can find comes from real life : the possibilities for human behavior are quite wild, however, in social life, people tend to restrict their possibilities, indeed to adapt to the external milieu (i.e. the orthogonal) which in turn reacts differently... In real life there is no winning strategy, i.e. the Princess's driver decides to drive with 2.3 grams of alcool because he thinks that no cop will dare to check him, but after all it is only wishful thinking ; there are even people who play losing strategies. By the way, in real games, behaviors play an essential role : for instance if you play against a beginner, you may try dirty tricks, that you would not dare against a master. I think that we should accept the existence of behaviors as a primitive fact, and logic as a way to analyze them, rather than behaviors a a way to comply with logical principles.

Coming back to the interpretation of plays as cut-elimination, one actually sees that only a finite terminal part of the proofs is visible at each step, which has two possible interpretations :

1. Either it is quite true that I am building the proof step by step ; after performing stage p, my opponent will test my last move, and I will answer with a new move. This is the viewpoint of interactive proof-search : if the starting configuration is $\vdash A \mathbin{\&} B$, II first chooses between $\vdash A^{\perp}$ and $\vdash B^{\perp}$; assuming that II chose the latter and that B is positive, then I will proceed with trying to prove $\vdash B$, and A is definitely forgotten.
2. Or the proof is known in advance, but it is made of nested boxes ; my opponent cannot understand my box at depth $p+1$ before I opened the box of depth p containing it. The original cut is between σ (ending with $\vdash A \mathbin{\&} B$) and τ (ending with $\vdash A^{\perp} \oplus B^{\perp}$), and II is asked to open his box : in the box there is $\vdash B^{\perp}$, and I obeys, *perinde ac cadaver*, and replaces his sequent with $\vdash B$... he must now in turn open a box etc.

These interpretations can as well represent (interactive) proof-search or cut-elimination ; by the way, for the first time, the two main readings of Gentzen's theorem are reconciled.

6 Conclusion

At which moment should we consider our program as achieved ? It is a question of mathematical taste, depending on the level of abstraction we are seeking. A first level of interpretation, purely combinatorial, is on its way, see [19] : what is still missing seems to be more technical than conceptual. It is enough to achieve our main goal, i.e. an *operational* interpretation of logic, where the basic rules are ≪ physical ≫ modifications of a geometrical structure. The old logical artefacts subsist as comments on the behavior of this process : *truth* is now the possibility of winning and *proof* the way to achieve it ; as to logical operations, they are no longer primarily defined as operations on truth, but as manipulations of games, which can be forgetfully interpreted on truth values. But the extant material is clearly too phenomenological (i.e. too explicit). Therefore there is still something essential to achieve after this step, some kind of revisited GoI.

We already mentioned that the earnest way to proceed is to accept that eventually linear logic itself could be modified, e.g. by the adjunction of the weakening rule ; after all the focus on linear logic is mainly due to the fact that it is the most general extant logical system, and that solving a problem in this case induces a solution for the other systems. But shall we find a unique interpretation, or are there several kinds of games that can solve our problem (inducing of course several logics), in other terms, shall we conclude to the existence of a unique logic ?

1. One could think that *cyclic linear logic* is based on a different paradigm, because of non-commutativity. But recent work of Ruet [27] and Abrusci, ending with a collaboration Abrusci-Ruet, see [3], has shown that cyclic linear logic is compatible with usual linear logic, i.e. that both a commutative and a non-commutative tensor may coexist. Surely this non-commutative logic will deserve another geometrical explanation, but eventually the unity will not be violated.
2. The existence of an alternative version of the exponentials, [16], might suggest that at some moment there might be a definite schism. It might also happen that only the *light exponentials* are eventually accepted... which would mean that the logical rules that are behind classical logic are definitely wrong. Very unlikely indeed, although exciting, since the alternative version (*light* linear logic) is polytime-complete.
3. The technical trick of hypersequentialization avoids the nightmare of commutation of rules, i.e. the possibility of non-alternating moves. But the asynchronous proof-nets were successful in the restricted multiplicative case... Hence something radical might explain the mismatch. We are perhaps explaining a *sequential* logic, and there might as well be a *parallel* logic —without temporality— ; these two logics would agree on the multiplicative part, and differ on other connectives. Of course such a hypothetical non-sequential logic would have non-sequential proofs etc. : we should not try to imitate what we are doing in the sequential case to investigate this possibility, and typically we could accept the Gustave function... I think that the ghost of an alternative *parallel* logic might vanish if we succeed to depart from the game intuition, in which a strict alternation of moves is so important.

Although we claim that the opposition syntax/semantics is obsolete, we are still focusing on soundness/completeness results. This can be read as the emergence of a new kind of semantics (something in between traditional syntax and semantics), a radically new viewpoint, in between (educated) Tarskism and (illiterate) gesticulation : whereas the former still believes in the pregnancy of the old distinction observer/world, the latter were never able to technically master it. In between Scylla and Charybdis lies the meaning of logical rules.

APPENDIX

The annexes expound known material, with a certain number of novel features. The most interesting are about time (Annex F), the phase semantics criterion which discrimitates between good and bad schemes (Annex C.4). and a baby-size realization of the program done in Annex E.7. The notion of CST of Annex D.5 is only introduced in view of the main discussion ; the presentation of completeness/incompleteness is rather different from the traditional approach, so it might be of some interest.

I had to make a choice as to formalism, since three logical systems (not to speak of non-commutative logic) should be presented. I decided to systematically present, when possible, the linear logic aspect, which after all makes sense, since linear logic is more general. For similar reasons, I decided to present a simplified syntax, exploiting the symmetry left/right of sequent calculus. As a consequence, intuitionistic logic (which does not enjoy the symmetry and needs a two-sided calculus) cannot be presented. Also the discussion in the main text of the symmetrization of intutionistic logic is not well-documented, since it speak of left and right parts of sequents, whereas our Annex anticipates on the result of the discussion... Apart from this, I think that the text is really self-contained, i.e. that the Appendix contains the crucial technical information in a sufficiently detailed way. (I didn't develop the C* algebraic version of GoI, just because it belongs to a spirit from which I am forced to depart for at least a while… but the interpretation of GoI in terms of wires, multiplexing, etc. is expounded in some details.)

A Completeness vs. incompleteness

A.1 Completeness

In logic, *completeness* has several technical meanings, which implement the idea that « *Nothing is missing* » : never forget it !
The basic result is due to Gödel (1930) and is about *classical predicate logic* : formulas are built from atomic propositions $P(t_1, \ldots, t_n)$ and their negations $\neg P(t_1, \ldots, t_n)$ by means of the connectives $\wedge, \vee$ and the first-order quantifiers $\forall x, \exists x$. And the theorem states that if A is closed and true in any model, then A is provable : classical logic catches all universal truths.
However when we say that A is closed, we only mean that it has no free variable… however the predicate symbols $P(\ldots)$ are in some sense variables. If we now move to *second order logic*, we can definitely close A by prefixing it with second order quantifications on predicates, e.g. $\forall P A$. Indeed we can speak of the (plain) truth of the universal closure of A, which is nothing but the truth of A for all possible P, i.e. the truth of A in all models. Completeness can therefore be written as

If A is a closed Π^1 formula, and A is true, then A is provable.

By Π^1 (resp. Σ^1), I mean that all second order quantifiers in A are universal (resp. existential), and by « closed » I mean definitely closed.

A.2 Incompleteness

Now, let us turn our attention towards incompleteness : Gödel produces one formula (in fact two formulas, since the theorem is in two parts : the first formula is « I am not provable », the second one is « The theory in which I am working is consistent »), and both express the unprovability of an explicit formula inside a well-defined system. Such an expression can be reduced (through painful and *ad hoc* encodings) to the form $\forall n \phi[n]$, quantification being on the set of integers (encoding all possible proofs) and ϕ being (morally) without quantifiers. Now the set of integers can be defined *à la* Dedekind, as the smallest set containing 0 and closed under successor :

$$n \in \mathbb{N} \quad \Leftrightarrow \quad \forall P((P(0) \wedge \forall x(P(x) \Rightarrow P(x+1)) \Rightarrow P(n))$$

which is Π^1 ; this formula, when used negatively (i.e. on the left hand of an implication) becomes Σ^1 : indeed, Gödel's formula is Σ^1. Now its property is to be true, but not provable. In other terms, in his famous result of 1931, Gödel proved a sort of converse to *completeness* :

Completeness fails outside Π^1.

Since Σ^1 and Π^1 are exchanged by negation, another way to look at completeness is to say that provability does not commute with negation : if A is Π^1 but *not* provable, we cannot conclude that $\neg A$ is provable. Certain mistakes of logic are due to the obstination at

producing a system in which provability commutes to negation : this has been a complete disaster, witness the fate of the unfortunate non-monotonic « logics » : there provability does commute to negation, but there is no longer any notion of proof... The theme of commutation of provability with other connectives, esp. disjunction and existence, had a different posterity, see below.
From the soundness of $\mathcal{L}$, I can deduce the consistency of $\mathcal{L}$ (soundness is always a technical variation on « A provable implies A true », hence as soon as something cannot be true, we get consistency). As a consequence, the soundness for $\mathcal{L}$ needs a « strong system », typically the soundness of second-order logic cannot be proved inside second-order logic.

A.3 About separation

The traditional way of proving completeness is by a fair proof-search : this means that we try to prove A by a systematic cumulative search, which is possible in the classical case. This is completely inefficient, but it works in theory : only two things might happen, either one ends with a proof, or the search never stops, in which case any infinite branch defines a countermodel. This vaguley looks like a duality, and it is, but it is not a real duality between proofs and countermodels, since they are exclusive of each other : either A has a proof, or it has a countermodel, but soundness makes it impossible to have both. In other terms, we cannot separate two proofs by countermodels, since duality abhors a vacuum. The notion of *test* developed in the main text is a less brutal notion of refutation, for which the opponent might coexist with the player.

B Sequent calculus

Sequent calculus was invented by Gentzen in 1934, [9, 11]. Before explaining the result and its consequences, we shall first define the (adaptation of the) calculus in the case of linear logic.

B.1 Linear sequent calculus

In order to present the calculus, we shall adopt the following notational conventions : formulas are written from literals p, q, r, $p^\perp$, $q^\perp$, $r^\perp$, etc., and constants $\mathbf{1}$, $\perp$, $\top$, $\mathbf{0}$ by means of the connectives $\otimes$, $\parr$, $\&$, $\oplus$ (binary), $!$, $?$ (unary), and the quantifiers $\forall x$, $\exists x$. Negation is *defined* by De Morgan equations, and linear implication is also a defined connective :

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathbf{1}^\perp & := & \perp & \perp^\perp & := & \mathbf{1} \\
\top^\perp & := & \mathbf{0} & \mathbf{0}^\perp & := & \top \\
(p)^\perp & := & p^\perp & (p^\perp)^\perp & := & p \\
(A \otimes B)^\perp & := & A^\perp \parr B^\perp & (A \parr B)^\perp & := & A^\perp \otimes B^\perp \\
(A \,\&\, B)^\perp & := & A^\perp \oplus B^\perp & (A \oplus B)^\perp & := & A^\perp \,\&\, B^\perp \\
(!A)^\perp & := & ?A^\perp & (?A)^\perp & := & !A^\perp \\
(\forall x\, A)^\perp & := & \exists x\, A^\perp & (\exists x\, A)^\perp & := & \forall x\, A^\perp
\end{array}$$

$$A \multimap B \;:=\; A^\perp \parr B$$

The connectives $\otimes$, $\parr$, $\multimap$, together with the neutral elements $\mathbf{1}$ (w.r.t. $\otimes$) and $\perp$ (w.r.t. $\parr$) are called *multiplicatives* ; the connectives $\&$ and $\oplus$, together with the neutral elements $\top$

(w.r.t. &) and $\mathbf{0}$ (w.r.t $\oplus$) are called *additives* ; the connectives ! and ? are called *exponentials*. The notation has been chosen for its mnemonic virtues : we can remember from the notation that $\otimes$ is multiplicative and conjunctive, with neutral $\mathbf{1}$, $\oplus$ is additive and disjunctive, with neutral $\mathbf{0}$, that $\parr$ is disjunctive with neutral $\perp$, and that & is conjunctive with neutral $\top$; the distributivity of $\otimes$ over $\oplus$ is also suggested by our notation ; this notation is related to *polarities*, see Annex F.

A *sequent* is an expression $\vdash \Delta$, where Δ is a finite sequence of formulas ; the intended meaning of $\vdash A_1, \ldots, A_n$ is $A_1 \parr \ldots \parr A_n$. We use the comma for concatenation ; very often we single out one formula in a sequent, e.g. $\vdash \Gamma, A$, and Γ is therefore seen as a *context*. The rules of the calculus are the following :

Identity / Negation

$$\frac{}{\vdash A, A^{\perp}}\ (\textit{identity}) \qquad\qquad \frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\textit{cut})$$

Structure

$$\frac{\vdash \Gamma}{\vdash \Gamma'}\ (\textit{exchange : } \Gamma' \textit{ is a permutation of } \Gamma)$$

Logic

$$\frac{}{\vdash \mathbf{1}}\ (\textit{one}) \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (\textit{false})$$

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ (\textit{times}) \qquad\qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (\textit{par})$$

$$\frac{}{\vdash \Gamma, \top}\ (\textit{true}) \qquad\qquad (\textit{no rule for zero})$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ (\textit{with}) \qquad\qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (\textit{left plus}) \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (\textit{right plus})$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (\textit{of course}) \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ (\textit{weakening})$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (\textit{dereliction}) \qquad\qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ (\textit{contraction})$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x\, A}\ (\textit{for all : } x \textit{ is not free in } \Gamma) \qquad\qquad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x\, A}\ (\textit{there is})$$

B.2 The Hauptsatz in the classical case

Classical logic has almost the same rules, except that ! and ? are replaced with the identity, i.e. erased, to the effect that *dereliction* and *of course* disappear, whereas *weakening* and *contraction* —which no longer mention ?— emigrate to the structural group. The presence of the structural rules causes a collapse, i.e. $\otimes$ and & which differ only through structural manipulations are identified (notation $\wedge$), and similarly $\parr = \oplus = \vee$.
The replacement of formulas by sequents $\vdash \Delta$ enabled Gentzen to almost reconcile two completely opposite approaches, i.e. intelligence and mechanism [31]. This is because the system is built around a specific rule, the *cut-rule*, which has an unbelievable duplicity :

- It basically expresses the transitivity of implication and therefore corresponds to 90 % of a real proof, which consists in putting together lemma after or above lemma... Remember that lemmas concentrate the use of intelligence, and that it may take years to find the right sequence of (perhaps easy) lemmas that will make your way to the theorem. The other rules are practically never used [32].
- The *Hauptsatz* paradoxically states that the cut-rule can be eliminated, i.e. that we can prove the same theorem without intelligence. One would then conclude that a machine can do it, and it is a matter of fact that the *subformula property* induces a drastic limitation of the search space. This limitation is so drastic that it is almost a *decision procedure* for logic : such a procedure cannot exist, but the search for cut-free proofs is efficient enough to serve as a basis for *logic programming*, popularized 15 years ago by the Japanese *5th generation program.*

The subformula property —which is also valid in the intuitionistic and linear cases— is indeed a completeness property : cut-free proofs od A can be localized inside a fixed system (the rules for the subformulas of A), i.e. nothing is missing. This is still the case if we extend the system into a second-order logic, as soon as we restrict to Π^1 formulas. But cut-elimination for Σ^1 formulas does not yield any subformula property, and various extensions of the calculus will therefore produce various sets of Σ^1 theorems : the system is no longer complete.

B.3 The reflection schema

An immediate by-product of the incompleteness theorem is the impossibility of defining a truth predicate in arithmetic, a result known as Tarski's theorem. However, there is no objection as to the definition of a bounded truth predicate, e.g.. a formula $Tr_{1540}[x]$, which represents, in a faithful way the truth of any formula A with less than 1540 logical connectives and quantifiers. Combining this with the *Hauptsatz* and the subformula property, Kreisel & Levy [20] were able to prove the following scheme inside Peano's arithmetic

$$Thm_{\mathbf{T}}(\ulcorner A[\overline{y}]\urcorner) \Rightarrow A[y]$$

Here $\mathbf{T}$ is classical predicate calculus (immediate extension : $\mathbf{T}$ is a finitely axiomatized subsystem of Peano's arithmetic), $A[x]$ is a given formula, depending on a free variable x, and $\ulcorner A[\overline{y}]\urcorner$ is the Gödel number (numerical encoding) of the closed formula A in which

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[31] If you want man and computer.

[32] The use of indirect arguments is the basic feature of thought : imagining that I still remember how to compute a product, then instead 61 × 21778945321, I will choose 21778945321 × 61, i.e. use commutativity as a lemma... and by the way the usual algorithm for multiplication in base 10 relies on a lemma.

the variable x has been replaced by the closed term $\overbrace{1+1+\ldots+1}^{y \text{ times}}$, and $Thm_{\mathbf{T}}$ represents provability in **T**. The proof consists in formally applying cut-elimination, then using a bounded truth predicate limited to subformulas of A, to formalize the trivial fact that logical rules preserve truth.
Application : Peano's arithmetic cannot be finitely axiomatized. In general the result is very helpful to produce inner formalizations.
Of course this illustrates the technical interest of something as flat as truth ; it is to be observed that we could do the same with intuitionistic logic, i.e. with Heyting's arithmetic, and for linear logic, should a reasonable linear arithmetic be introduced. But it does not work for the arithmetic of *Broccoli*, since the *Broccoli* axiom does not enjoy cut-elimination : in other terms truth seems only to make sense because of cut-elimination and we are back to our basic claim of an inner completeness expressed by the subformula property rather than an external completeness based on preservation of truth.

B.4 The Hauptsatz in the intuitionistic case

The original version of sequent calculus uses double-sided sequents $\Gamma \vdash \Delta$; in the classical and linear cases, the perfect symmetry between left and right makes it unecessary to use both sides : instead of $\Gamma \vdash \Delta$, write $\vdash \Gamma^{\perp}, \Delta$, with no noticeable change, but for drastic slimming of the calculus. However this cannot be done for intuitionistic logic, which is based on asymmetric sequents $\Gamma \vdash A$ (which mean that the conjunction of the formulas of Γ implies A). Weakening and contraction are permitted on the left, but not on the right, since the limitation to one formula makes them obsolete. This limitation enables one to formulate the *disjunction property*, which is specific of intuitionistic —and now linear— logic : if I prove a disjunction $A \vee B$, then I prove A or I prove B. In fact, although the result can be expected from Brouwer's considerations concerning the excluded middle $A \vee \neg A$, it is again a summit of duplicity :

1. In real life, nobody (except perhaps in cryptography) would state $A \vee B$ if he has a proof of B... the proposition is empty from the viewpoint of sociology : all proofs of $A \vee B$ are indirect, i.e. we state the disjunction for want of a more precise statement.

2. But if we apply the famous algorithm underlying the proof of the *Hauptsatz*, we end with one of those unbelievable cut-free proofs, which explicitly contains a proof of A or a proof of B.

Technically speaking, the disjunction property in —say linear logic— is obtain as follows : in a cut-free proof of $A \oplus B$ the last rule must be one of the $\oplus$-rules. Intuitionistic logic managed to make provability commute with disjunction... linear logic too, but only for disjunction $\oplus$, $\parr$ behaving more like a classical disjunction. By the way, the existential quantifier has a similar existence property [33].
This phenomenon explains why intuitionistic logic is *constructive* : proofs induce algorithms that can construct actual answers. In fact this constructivity can be developed into a full algorithmic interpretation. Typically the choice between A and B can serve to encode a Boolean answer (yes/no). Now, if I am given a proof π of $C \vdash A \vee B$, and if I can encode a certain datum a by a proof μ_a of $\vdash C$, then the *Hauptsatz* applied to the proof of $\vdash A \vee B$

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[33] It is not always the case that somebody who knows that $\phi[37]$ would never state $\exists n \phi[n]$, think of *regular prime numbers* : the primary fact is that not all prime numbers are regular, i.e. that methods based on regularity will not apply in general.

obtained by *cutting* π with λ_a induces a Boolean output $\pi(\lambda_a)$. The *Hauptsatz* is indeed the starting point of an integrated paradigm of functional programming.
Gentzen's result has therefore two posterities, that can be summarized as proof-search (subformula property) and proof-normalization (exploitation of the disjunction property by the algorithm of cut-elimination) ; but only proof-normalization has a sophisticated theory [34].

B.5 Non-commutative logic

Cyclic linear logic is obtained by restricting the exchange rule to circular permutations, keeping the rules as they are. But one should careful when defining negation, typically $(A \otimes B)^\perp = B^\perp \parr A^\perp$, which is reminiscent of $(uv)^* = v^*u^*$. Exponentials cannot be accommodated in this system, since commutativity plays an essential role here
($!(A \mathbin{\&} B) \simeq !A \otimes !B$ forces some commutativity). The work in progress of Abrusci and Ruet [3] enables one to reconcile commutativity and non-commutativity and should therefore accommodate exponentials.

C Phase semantics

C.1 Phase spaces

A *phase space* is a pair $(M, \perp)$, where M is a commutative monoid (usually written multiplicatively) and $\perp$ is a subset of M. Given two subsets X and Y of M, one can define $X \multimap Y := \{m \in M \,;\, mX \subset Y\}$. In particular, we can define for each subset X of M its *orthogonal* $X^\perp := X \multimap \perp$. A *fact* is any subset of M equal to its biorthogonal, or equivalently any subset of the form $Y^\perp$. It is immediate that $X \multimap Y$ is a fact as soon as Y is a fact.

C.2 Connectives

There are some basic ways of constructing facts, e.g. basic facts such as $\perp = \mathbf{1}^\perp$ or operations sending facts to facts, typically $X^\perp$, called (linear) negation, or $X \multimap Y$, called linear implication. The most important ones are listed below :

1. **times** : $X \otimes Y := (X.Y)^{\perp\perp} = \{mn \,;\, m \in X \wedge n \in Y\}^{\perp\perp}$
2. **par** : $X \parr Y := (X^\perp.Y^\perp)^\perp$
3. **one** : $\mathbf{1} := \{1\}^{\perp\perp}$, where 1 is the neutral element of M
4. **plus** : $X \oplus Y := (X \cup Y)^{\perp\perp}$
5. **with** : $X \mathbin{\&} Y := X \cap Y$
6. **zero** : $\mathbf{0} := \emptyset^{\perp\perp}$
7. **true** : $\top := M$
8. **of course** : $!X := (X \cap \mathcal{I})^{\perp\perp}$, where $\mathcal{I}$ is the set of idempotents of M which belong to $\mathbf{1}$
9. **why not** : $?X := (X^\perp \cap \mathcal{I})^\perp$

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[34] One of the main features of our approach is to reconcile cut-elimination with (interactive) proof-search.

The interpretation of the exponentials ! and ? is an improvement of the original definition of [12] which was *ad hoc* : $X \cap \mathcal{I}$ replaces a sort of topological interior X°.
Propositional linear logic is the logical system whose connectives are precisely the operations just introduced.

C.3 Soundness and completeness

Formulas are interpreted by facts, and a fact will be considered as true when it contains the neutral element 1. Then, it is easily seen that this semantics is sound and complete :

Theorem 1

A formula A of linear logic is provable iff for any interpretation (involving a phase space $(M, \perp)$), the interpretation $A^\bullet$ of A contains the neutral element 1.

Proof. — Soundness is proved by a straightforward induction. Completeness involves the building of a specific phase space. In fact we can take as M the monoid of contexts (i.e. multisets of formulas [35]), whose neutral element is the empty context, and we define $\perp := \{\Gamma \ ; \ \vdash \ \Gamma \ \text{ provable}\}$. The proof proceeds by showing that the sets $A^\bullet := \{\Gamma \ ; \ \vdash \ \Gamma, A \ \text{ provable}\}$ are facts. More precisely, one can prove (using the identity group) that $A^{\perp\bullet} = A^{\bullet\perp}$. It is then quite easy to prove that $A^\bullet$ is indeed the value of A in a given model : this amounts to prove commutations of the style $(A \otimes B)^\bullet = A^\bullet \otimes B^\bullet$ (these proofs are simplified by the fact that in any De Morgan pair $\otimes / \parr, \oplus / \&, ! / ?$ one commutation implies the other, hence we can choose the friendlier commutation). Therefore, if $1 \in A^\bullet$, it follows that $\vdash \ A$ is provable. $\square$

C.4 Naturality

We shall submit three additional principles, namely weakening, contraction, and *Broccoli* to the same test, and see what happens. Indeed, our only commitment is to fire a particular consistent scheme, so we shall modify *Broccoli* into a stronger version in which $\lozenge = \otimes$, and which is sometimes called « reverse contraction ».
The test will be as follows : we shall consider those phase spaces enjoying the three principles when the parameters are atomic facts $a^{\perp\perp}$ and see whether or not the same property holds for any fact [36].

C.4.1 Weakening

Weakening is $A \otimes B \multimap A$; we impose it for atomic facts $a^{\perp\perp}, b^{\perp\perp}$, i.e. we restrict to those models such that $1 \in a^{\perp\perp} \otimes b^{\perp\perp} \multimap a^{\perp\perp}$. Now $1^{\perp\perp}$ is easily shown to be neutral w.r.t. tensor, hence we get $1 \in b^{\perp\perp} \multimap 1^{\perp\perp}$, which is the same as $b^{\perp\perp} \subset 1^{\perp\perp}$, which in turn is the same as $\perp \subset b^{\perp}$. In other terms our condition is equivalent to saying that $\perp$ is an ideal, i.e. $a \in \perp \Rightarrow ab \in \perp$. Now consider general fact A, B ; if $a \in A, b \in B, c \in B^{\perp}$, then $bc \in \perp$, and since $\perp$ is an ideal, $abc \in \perp$. We just proved that $A.B \subset B^{\perp\perp}$, hence $A \otimes B = (A.B)^{\perp\perp} \subset B^{\perp\perp} = B$. Weakening makes it !

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[35] We ignore the multiplicity of formulas $?\Gamma$, so that $\mathcal{I}$ is the set of contexts $?\Gamma$

[36] Requiring a property for all atomic facts is a first-order statement, whereas its analogue for arbitrary facts is second-order. The test is a reduction of second-order to first-order which is in accordance with the general spirit of the subformula property, and more generally of cut-elimination. By the way it would be interesting to make the connection between the test and cut-elimination more precise.

C.4.2 Contraction

Contraction is $A \multimap A \otimes A$; we impose it for atomic facts $a^{\perp\perp}$, i.e. we restrict to those models such that $1 \in a^{\perp\perp} \multimap a^{\perp\perp} \otimes a^{\perp\perp}$. This is easily simplified into $a \in a^{\perp\perp} \otimes a^{\perp\perp}$. Now an essential property, see proposition 3 is that $X^{\perp\perp} \otimes Y^{\perp\perp} \subset (X.Y)^{\perp\perp}$, and we can simplify our equation into $a \in a^{2\perp\perp}$, which is the same as $a^{2\perp} \subset a^{\perp}$: we eventually find the condition $a^2 b \in \perp \Rightarrow ab \in \perp$. Now consider a general fact A ; if $a \in A$, then $a^2 \in A.A$. But if $b \in (A.A)^{\perp}$, $a^2 b \in \perp$, so we can conclude from the specificities of $\perp$ that $ab \in \perp$: this proves that $a \in (A.A)^{\perp\perp} = A \otimes A$. Hence $A \subset A \otimes A$ and contraction makes it ! By the way the combination weakening + contraction —i.e. classical logic— makes it too.

C.4.3 Broccoli

Broccoli is $A \otimes A \multimap A$; we impose it for atomic facts $a^{\perp\perp}$, i.e. we restrict to those models such that $1 \in a^{\perp\perp} \otimes a^{\perp\perp} \multimap a^{\perp\perp}$. This is immediately transformed into (again use proposition 3) $a^{2\perp\perp} \subset a^{\perp\perp}$, i.e. into $a^{\perp} \subset a^{2\perp}$, which is the principle $ab \in \perp \Rightarrow a^2 b \in \perp$. Now consider a general fact A ; we would like to prove that $A \otimes A \subset A$, which is equivalent to $A^{\perp} \subset (A.A)^{\perp}$. In order to achieve this, we must show that if b is orthogonal to all $a \in A$ then it is orthogonal to all products $a.a'$, with $a, a' \in A$. For this one would need something like $ab \in \perp \wedge a'b \in \perp \Rightarrow aa'b \in \perp$, but we only got it when $a = a'$. From this it is easy to build a counterexample, and this is why *Broccoli* does not make it.

D Coherent spaces

D.1 Coherent spaces

Definition 1

> *A* coherent space *is a reflexive undirected graph. In other terms it consists of a set* $|X|$ *of* atoms *together with a compatibility or* coherence *relation between atoms, noted* $x \mathrel{\overset{\frown}{\smile}} y$ *or* $x \mathrel{\overset{\frown}{\smile}} y\ [\mathrm{mod}\ X]$ *if there is any ambiguity as to* X.
>
> *A* clique *a in X (notation $a \sqsubset X$) is a subset a of X made of pairwise coherent atoms :* $a \sqsubset X$ *iff* $\forall x \forall y\ (x \in a \wedge y \in a \Rightarrow x \mathrel{\overset{\frown}{\smile}} y)$.

Besides coherence we can also introduce

- *strict coherence* : $x \frown y$ iff $x \mathrel{\overset{\frown}{\smile}} y$ and $x \neq y$,
- *incoherence* : $x \asymp y$ iff $\neg(x \frown y)$,
- *strict incoherence* : $x \smile y$ iff $\neg(x \mathrel{\overset{\frown}{\smile}} y)$.

Any of these four relations can serve as a definition of coherent space. Observe fact that $\asymp$ is the negation of $\frown$ and not of $\mathrel{\overset{\frown}{\smile}}$; this is due to reflexivity.

Definition 2

> *Given a coherent space X, its* linear negation $X^{\perp}$ *is defined by*
>
> - $|X^{\perp}| = |X|$,
> - $x \mathrel{\overset{\frown}{\smile}} y\ [\mathrm{mod}\ X^{\perp}]$ *iff* $x \asymp y\ [\mathrm{mod}\ X]$.

In other terms, linear negation is nothing but the exchange of coherence and incoherence ; It is obviously involutive : $X^{\perp\perp} = X$.

Definition 3

> *Given two coherent spaces X and Y, the* multiplicative connectives $\otimes$, $\parr$, $\multimap$ *define a new coherent space Z with $|Z| = |X| \otimes |Y|$; coherence is defined by*
>
> - $(x,y) \overset{\frown}{\smile} (x',y')$ $[\mathrm{mod}\ X \otimes Y]$ *iff*
> $x \overset{\frown}{\smile} x'$ $[\mathrm{mod}\ X]$ *and* $y \overset{\frown}{\smile} y'$ $[\mathrm{mod}\ Y]$,
> - $(x,y) \frown (x',y')$ $[\mathrm{mod}\ X \parr Y]$ *iff*
> $x \frown x'$ $[\mathrm{mod}\ X]$ *or* $y \frown y'$ $[\mathrm{mod}\ Y]$,
> - $(x,y) \frown (x',y')$ $[\mathrm{mod}\ X \multimap Y]$ *iff*
> $x \overset{\frown}{\smile} x'$ $[\mathrm{mod}\ X]$ *implies* $y \frown y'$ $[\mathrm{mod}\ Y]$.

Observe that $\otimes$ is defined in terms of $\overset{\frown}{\smile}$ but $\parr$ and $\multimap$ in terms of $\frown$. A lot of useful isomorphisms can be obtained

1. De Morgan equalities : $(X \otimes Y)^\perp = X^\perp \parr Y^\perp$; $(X \parr Y)^\perp = X^\perp \otimes Y^\perp$; $X \multimap Y = X^\perp \parr Y$;
2. commutativity isomorphisms : $X \otimes Y \simeq Y \otimes X$; $X \parr Y \simeq Y \parr X$; $X \multimap Y \simeq Y^\perp \multimap X^\perp$;
3. associativity isomorphisms : $X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$; $X \parr (Y \parr Z) \simeq (X \parr Y) \parr Z$; $X \multimap (Y \multimap Z) \simeq (X \otimes Y) \multimap Z$; $X \multimap (Y \parr Z) \simeq (X \multimap Y) \parr Z$.

Definition 4

> *Up to isomorphism there is a unique coherent space consisting of one atom 0, this space is self dual, i.e. equal to its linear negation. However the algebraic isomorphism between this space and its dual is logically meaningless, and we shall, depending on the context, use the notation $\mathbf{1}$ or the notation $\perp$ for this space, with the convention that $\mathbf{1}^\perp = \perp$, $\perp^\perp = \mathbf{1}$.*

This space is neutral w.r.t. multiplicatives, namely $X \otimes \mathbf{1} \simeq X$, $X \parr \perp \simeq X$, $\mathbf{1} \multimap X \simeq X$, $X \multimap \perp \simeq X^\perp$. This notational distinction is mere preciosity ; one of the main drawbacks of denotational semantics is that it interprets logically irrelevant properties . . . but nobody is perfect.

Definition 5

> *Given two coherent spaces X and Y the* additive connectives $\&$ *and* $\oplus$, *define a new coherent space Z with $|Z| = |X| + |Y|$ $(= |X| \otimes \{0\} \cup |Y| \otimes \{1\})$; coherence is defined by*
>
> - $(x,0) \overset{\frown}{\smile} (x',0)$ $[\mathrm{mod}\ Z]$ *iff* $x \overset{\frown}{\smile} x'$ $[\mathrm{mod}\ X]$,
> - $(y,1) \overset{\frown}{\smile} (y',1)$ $[\mathrm{mod}\ Z]$ *iff* $y \overset{\frown}{\smile} y'$ $[\mathrm{mod}\ Y]$,
> - $(x,0) \frown (y,1)$ $[\mathrm{mod}\ X \,\&\, Y]$,
> - $(x,0) \smile (y,1)$ $[\mathrm{mod}\ X \oplus Y]$.

A lot of useful isomorphisms are immediately obtained :

- De Morgan equalities : $(X \,\&\, Y)^\perp = X^\perp \oplus Y^\perp$; $(X \oplus Y)^\perp = X^\perp \,\&\, Y^\perp$;
- commutativity isomorphisms : $X \,\&\, Y \simeq Y \,\&\, X$; $X \oplus Y \simeq Y \oplus X$;
- associativity isomorphisms : $X \,\&\, (Y \,\&\, Z) \simeq (X \,\&\, Y) \,\&\, Z$; $X \oplus (Y \oplus Z) \simeq (X \oplus Y) \oplus Z$;

- distributivity isomorphisms : $X \otimes (Y \oplus Z) \simeq (X \otimes Y) \oplus (X \otimes Z)$; $X ⅋ (Y \& Z) \simeq (X ⅋ Y) \& (X ⅋ Z)$; $X \multimap (Y \& Z) \simeq (X \multimap Y) \& (X \multimap Z)$; $(X \oplus Y) \multimap Z \simeq (X \multimap Z) \& (Y \multimap Z)$.

The other distributivities fail ; for instance $X \otimes (Y \& Z)$ is not isomorphic to $(X \otimes Y) \& (X \otimes Z)$.

Definition 6

There is a unique coherent space with no atom. Although this space is also self dual, we shall use distinct notations for it and its negation : $\top$ *and* $\mathbf{0}$.

These spaces are neutral w.r.t. additives : $X \oplus \mathbf{0} \simeq X$, $X \& \top \simeq X$, and absorbing w.r.t. multiplicatives $X \otimes \mathbf{0} \simeq \mathbf{0}$, $X ⅋ \top \simeq \top$, $\mathbf{0} \multimap X \simeq \top$, $X \multimap \top \simeq \top$.

Definition 7

If X *is a coherent space, then* $!X$ *is defined as follows :*

- $|!X| = X_{fin}$, *where* X_{fin} *is the set of all finite cliques of* X
- $a \frown b \Leftrightarrow a \cup b \sqsubset X$

The space $?X$ *is defined as* $(!(X^{\perp}))^{\perp}$, *so as to satisfy the De Morgan equalities :* $(!X)^{\perp} = ?(X^{\perp})$; $(?X)^{\perp} = !(X^{\perp})$.

Among remarkable isomorphisms let us mention the exponentiation isomorphisms :
$!(X \& Y) \simeq (!X) \otimes (!Y)$; $?(X \oplus Y) \simeq (?X) ⅋ (?Y)$, together with the "particular cases" $!\top \simeq \mathbf{1}$; $?\mathbf{0} \simeq \perp$.

D.2 Interpretation of linear logic

The main notion of linear logic is that of a linear map

Definition 8

Let X *and* Y *be coherent spaces ; a* linear map *from* X *to* Y *consists in a function* F *such that*

1. *if* $a \sqsubset X$ *then* $F(a) \sqsubset Y$,
2. *if* $\bigcup b_i = a \sqsubset X$ *then* $F(a) = \bigcup F(b_i)$,
3. *if* $a \cup b \sqsubset X$, *then* $F(a \cap b) = F(a) \cap F(b)$.

These conditions can be rephrased as the preservation of disjoint unions.

Proposition 1

There is a 1-1 correspondence between linear maps from X *to* Y *and cliques in* $X \multimap Y$ *; more precisely*

- *to any linear* F *from* X *to* Y, *associate* $\mathrm{Tr}(F) \sqsubset X \multimap Y$ *(the* trace *of* F*)*

$$\mathrm{Tr}(F) = \{(x, y) \ ; \ y \in F(\{x\})\},$$

- *to any* $A \sqsubset X \multimap Y$ *associate a linear function* $A(\cdot)$ *from* X *to* Y

$$\text{if } a \sqsubset X, \text{ then } A(a) = \{y \ ; \ \exists x \in a \, (x, y) \in A\}.$$

Proof. — The proofs that $\mathrm{Tr}(A(\cdot)) = A$ and $\mathrm{Tr}(F)(\cdot) = F$ are left to the reader. □
Since $X \multimap Y \simeq Y^\perp \multimap X^\perp$, a linear map from X to Y induces an *adjoint* map from $Y^\perp$ to $X^\perp$. Equivalently, a clique in the coherent space $X ⅋ Y$ can be seen either as a linear function from $X^\perp$ into Y, or as a linear function from $Y^\perp$ into X. More generally a clique in $X_1 ⅋ \ldots ⅋ X_n$ can be seen as a multilinear map from ${X_1}^\perp, \ldots, \widehat{X_i}, \ldots, {X_n}^\perp$ to X_i by deciding to *focus* on some X_i.
Now a proof π of a sequent $\vdash A_1, \ldots, A_n$ will be interpreted as a clique in the space $|A_1| ⅋ \ldots ⅋ |A_n|$, and therefore, as a a multilinear function, *modulo* an appropriate focalization. The fact that we can choose the focus is very useful : we always focus on the formula which is introduced by the rule, and we are basically led to interpret logical rules as pointwise operations on multilinear functions, for instance :

Definition 9

1. *If $a \sqsubset X$, $b \sqsubset Y$, then $a \otimes b = \{(x,y); x \in a, y \in b\}$ is a clique in $X \otimes Y$. This clique is the essential ingredient in the rule of $\otimes$.*
2. *With an appropriate focalization, the rule of ⅋ reduces to the formation of the trace of a linear function.*
3. *If $a \sqsubset X$, $b \sqsubset Y$, then $l(a) = \{(x,0); x \in a\}$ and $r(b) = \{(y,1); y \in b\}$ is a clique in $X \oplus Y$. These cliques are the essential ingredients in the rules of $\oplus$.*
4. *If $a \sqsubset X$, $b \sqsubset Y$, then then $a \,\&\, b = \{(x,0); x \in a\} \cup \{(y,1); y \in b\}$ is a clique in $X \,\&\, Y$. This clique is the essential ingredient in the rule of $\&$.*
5. *If $a \sqsubset X$, $!a = \{u; u \in X_{fin}, u \subset a\}$ is a clique in $!X$. This clique is the essential ingredients in the rule of $!$, but one must be careful.*

Modulo focalization, the identity axiom corresponds to the identity map, and the cut-rule to composition. The main result is the invariance of the interpretation under cut-elimination, which the moral content of proposition 1.

D.3 The bridge with intuitionism

Coherent semantics was originally developed as a semantics of intuitionistic logic.

Definition 10

Let X and Y be coherent spaces ; a stable *map from X to Y is a function F such that*

1. *if $a \sqsubset X$ then $F(a) \sqsubset Y$,*
2. *assume that $a = \bigcup b_i$, where b_i is directed with respect to inclusion, then $F(a) = \bigcup F(b_i)$,*
3. *if $a \cup b \sqsubset X$, then $F(a \cap b) = F(a) \cap F(b)$.*

Stability i.e. condition (3), is due to Berry, although he formulated it for more complicated spaces : indeed the parallel or (see subsection 5.5.4) is not a stable function.

Definition 11

Let X and Y be coherent spaces ; then we define the coherent space $X \Rightarrow Y$ as follows :

- $|X \Rightarrow Y| = X_{fin} \times |Y|$,
- *$(a, y) \supset\!\!\!\subset (a', y')$ iff (1) and (2) hold :*
  1. $a \cup a' \sqsubset X \;\Rightarrow\; y \supset\!\!\!\subset y'$,
  2. $a \cup a' \sqsubset X \,\wedge\, a \neq a' \;\Rightarrow\; y \frown y'$.

Proposition 2

There is a 1-1 correspondence between stable maps from X to Y and cliques in $X \Rightarrow Y$; more precisely

1. *to any stable F from X to Y, associate* $\mathrm{Tr}(F) \sqsubset X \Rightarrow Y$

$$\mathrm{Tr}(F) = \{(a, y) \; ; \; a \sqsubset X \;\; \wedge \;\; y \in F(a) \;\; \wedge \forall a' \subset a \, (y \in F(a') \Rightarrow a' = a)\}$$

2. *to any $A \sqsubset X \Rightarrow Y$, associate a stable function $A(\,\cdot\,)$ from X to Y*

$$\textit{if } a \sqsubset X, \; \textit{then } A(a) = \{y \; ; \; \exists b \subset a \, ((b, y) \in A)\}.$$

Proof. — The essential ingredient is the normal form theorem below. □

Theorem 2

Let F be a stable function from X to Y, let $a \sqsubset X$, let $y \in F(a)$; then

1. *there exists $a_0 \subset a$, a_0 finite, such that $y \in F(a_0)$,*
2. *if a_0 is chosen minimal w.r.t. inclusion, then it is unique.*

Proof. — (1) follows from $a = \bigcup a_i$, the directed union of its finite subsets ; $z \in F(\bigcup a_i) = \bigcup F(a_i)$ hence $z \in F(a_i)$ for some i.

(2) : given two solutions a_0, a_1 included in a, we get $z \in F(a_0) \cap F(a_1) = F(a_0 \cap a_1)$; if a_0 is minimal w.r.t. inclusion, this forces $a_0 \cap a_1 = a_0$, hence $a_0 \subset a_1$. □

This establishes the basic bridge with linear logic, since $X \Rightarrow Y$ is strictly the same thing as $!X \multimap Y$. Linear logic was indeed extracted from this factorization —an illustration that semantics may have a real feedback on syntax. In fact one can translate intuitionistic logic into linear logic as follows :

$$\begin{aligned}
p^* &:= p \quad (p \text{ atomic}), \\
(A \Rightarrow B)^* &:= !A^* \multimap B^*, \\
(A \wedge B)^* &:= A^* \,\&\, B^*, \\
(\forall x\, A)^* &:= \forall x\, A^*, \\
(A \vee B)^* &:= !A^* \oplus !B^*, \\
(\exists x\, A)^* &:= \exists x\, !A^*, \\
(\neg A)^* &:= !A^* \multimap \mathbf{0}.
\end{aligned}$$

and prove the following result: $\Gamma \vdash A$ is intuitionistically provable iff $!\Gamma^* \vdash A^*$ (i.e. $\vdash ?\Gamma^{*\perp}, A^*$) is linearly provable. The possibility of such a faithful translation is of course a major evidence for linear logic, since it links it with intuitionistic logic in a strong sense. In particular linear logic can *at least* be accepted as a way of analyzing intuitionistic logic. As an exercise, one should try to prove the isomorphism $(X \wedge Y) \Rightarrow Z \simeq X \Rightarrow (Y \Rightarrow Z)$ by translating everything in linear logic and applying the isomorphisms listed in the opening subsection.

D.4 Exponentials and comonoids

Let us come back to exponentials ; the space $!X$ is equipped with two maps :

$$c \in !X \multimap (!X \otimes !X) \qquad\qquad w \in !X \multimap 1$$

corresponding to contraction and weakening. We can see these two maps as defining a structure of *comonoid* : intuitively this means the contraction map behaves like a commutative/associative law and that the weakening map behaves like its neutral element. The only difference with a usual monoid is that the arrows are in the wrong direction. A comonoid is therefore a triple (X, c, w) satisfying conditions of (co)-associativity, commutativity and neutrality. $?X$, which is the dual of a comonoid, is not a monoid (since $\otimes$ dualizes as $\parr$), we say that it is a *Par-monoid.*

D.5 Totality

It is possible to prove a real completeness theorem w.r.t. coherent spaces, see [18]. We shall here present an oversimplification of this result, by means of *coherent spaces with totality* (CST), which are well-suited for our main discussion.

Definition 12

Let X be a coherent space ; a clique $a \sqsubset X$ is said to be orthogonal *to an anticlique $b \sqsubset Y$ when they do intersect (in which case the intersection consist of one point) ; notation $a \perp b$. A CST is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a set of cliques of X equal to its biorthogonal. In particular $(X^{\perp}, \mathcal{T}^{\perp})$ is a CST.*

All constructions of coherent spaces can be accommodated into constructions of CST : the underlying coherent spaces are as above.

Definition 13

1. *There is only one total element in* **1***, namely the clique* $\{0\}$.
2. *Using definition 9, the tensor product of* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *is* $(X \otimes Y, (\mathcal{T} \otimes \mathcal{U})^{\perp\perp})$.
3. *There is no total element in* **0**.
4. *Using definition 9, the direct sum of* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *is* $(X \oplus Y, (l(\mathcal{T}) \cup r(\mathcal{U}))^{\perp\perp})$.

The other operations can be obtained by orthogonality ; observe that :

1. There is only one total element in $\perp$, namely the clique $\{0\}$.

2. A clique F in a linear implication is total when it maps total elements to total elements.

3. A clique in a « & » is total when both projections are total.

It is not difficult to now prove that all cliques constructed in the interpretation of linear logic are total (again, focalize !). However there are total cliques which do not come from a proof, e.g. the unique clique of $\perp$. Those cliques should therefore correspond to paraproofs, but how to distinguish proofs from general paraproofs ? The paper [18] parametrizes the construction, coherent spaces being replaced by « free modules over a Par-monoid $\mathbb{P}$ », with the result that orthogonality yields a clique in $\mathbb{P}$, and there might be so many of these cliques that everything works fine. Indeed one can see $\vdash \Gamma, A$ as something like $\gamma.a$, the context γ being treated like a scalar, and the cut-rule as a bilinear form. But this result, although non-trivial, had no feedback on anything, and we should admit that it might still contain

some gesticulation, mainly due to this awfully abstract « Par-monoid ».
A more naive answer (that we only mention for the sake of the main discussion) would be to attach to each atom a the *result* $r(a)$ of the corresponding play, in particular, when forming the negation, the result n would be changed into $1-n$, winning corresponding to the value 1. This incorrect idea has the good taste to propose a full solution to our problem, namely :

- Game = CST
- Play = atom
- Strategy = total clique
- Composition of strategies : the unique atom in $\sigma \cap \tau$
- Output of the play = $r(\sigma \cap \tau)$

This pattern is technically incorrect, as we can see from the main discussion ; but it has the immense virtue to be simple, and —hopefully— not too far from the solution.

E Proof-nets

Proof-nets are a sort of parallel syntax primarily developed in the pure multiplicative case.

E.1 Proof-structures

Definition 14

A link *L is an expression*

$$\frac{P_1, \ldots, P_n}{Q_1, \ldots, Q_m} L$$

involving n formulas (the premises *of L) $P_1, \ldots, P_n$ and m formulas (the* conclusions *of L) $Q_1, \ldots, Q_m$; we sall use the following links :*

ID −links	:	0 premise			2 conclusions :		$A, A^\perp$
CUT −links	:	2 premises	:	$A, A^\perp$	0 conclusion		
$\otimes$ −links	:	2 premises	:	A, B	1 conclusion	:	$A \otimes B$
$⅋$ −links	:	2 premises	:	A, B	1 conclusion	:	$A ⅋ B$

The premises of $\otimes$, $⅋$-links are ordered : this means that we can distinguish a left premise (here A) and a right premise (here B). On the other hand the premises of a CUT-link and the conclusions of an ID-link are unordered.

(*axiom link*)

(*cut link*)

$$\frac{A \quad B}{A \otimes B} \ (\textit{times link}) \qquad \frac{A \quad B}{A ⅋ B} \ (\textit{par link})$$

Remark. — Due to multiple occurrences, one should never speak of formulas, but of *occurrences*, which is extremely awkward. We adopt once for all the convention that all our formulas are distinct (for instance by adding extra indices). In particular $ID, \otimes$, $⅋$-links are determined by their conclusion(s), and a CUT-link is determined by its premises.

Definition 15

A proof-structure Θ *consists of :*

- *A set of formulas (see the previous remark) ;*
- *A set of links ; each of these links takes its premise(s) and conclusion(s) among the formulas of* Θ.

such that each formula is the the premise of at most *one link and the conclusion of exactly one link ; the formulas which are not premises of some link are called the* conclusions *of* Θ.

E.2 Sequentialization

To any proof of $\vdash \Gamma$ in linear (multiplicative) sequent calculus we associate a *proof-structure* with as conclusions the formulas of Γ. More precisely :

1. To the identity axiom associate an axiom link.

2. Do not interpret the exchange rule (this rule does not affect conclusions ; however, if we insist on writing a proof-structure on a plane, the effect of the rule can be seen as introducing *crossings* between axiom links ; planar proof-structures will therefore correspond to proofs in some non-commutative variants of linear logic).

3. If a proof-structure Θ ending with Γ, A and B has been associated to a proof π of $\vdash \Gamma, A, B$ and if one now applies a "par" rule to this proof to get a proof π' of $\vdash \Gamma, A \parr B$, then the structure Θ' associated to π' will be obtained from Θ by linking A and B via a *par* link : therefore A and B are no longer conclusions, and a new conclusion $A \parr B$ is created.

4. If π_1 is a proof of $\vdash \Gamma, A$ and π_2 is a proof of $\vdash B, \Delta$ to which proof-structures Θ_1 and Θ_2 have been associated, then the proof π' obtained from π_1 and π_2 by means of a times rule is interpreted by means of the proof structure Θ obtained from Θ_1 and Θ_2 by linking A and B together via a *times* link. Therefore A and B are no longer conclusions and a new conclusion $A \otimes B$ is created.

5. If π_1 is a proof of $\vdash \Gamma, A$ and π_2 is a proof of $\vdash A^{\perp}, \Delta$ to which proof-structures Θ_1 and Θ_2 have been associated, then the proof π' obtained from π_1 and π_2 by means of a cut rule is interpreted by means of the proof structure Θ obtained from Θ_1 and Θ_2 by linking A and $A^{\perp}$ together via a *cut* link. Therefore A and $A^{\perp}$ are no longer conclusions.

Definition 16

A proof-structure Θ *is* sequentializable *when it is associated to a proof of sequent calculus. This proof is called a* sequentialization *of* Θ *and is not unique : permutations of rules will produce the same structure.*

Anyway, the main problem is to find a *sequentialisation* theorem ; this means to give an intrinsic characterization of *sequentializable* proof-structures.

E.3 Proof-nets

Definition 17

A switching *$\mathcal{S}$ of a proof-structure Θ consists in the selection of a choice $\mathcal{S}(L) \in \{l, r\}$ for all ⅋-links of $\varphi_{\mathcal{S}}(\Theta)$.*

Definition 18

Let $\mathcal{S}$ be a switching of a proof-structure Θ ; we define the graph $\Theta_{\mathcal{S}}$ as follows :

- *The vertices of $\Theta_{\mathcal{S}}$ are the formulas of Θ ;*
- *For all ID-links of $\varphi_{\mathcal{S}}(\Theta)$, we draw an edge between the conclusions ;*
- *For all CUT-links of $\varphi_{\mathcal{S}}(\Theta)$, we draw an edge between the premises ;*
- *For all $\otimes$-links of $\varphi_{\mathcal{S}}(\Theta)$, we draw an edge between the left premise and the conclusion, and between the right premise and the conclusion ;*
- *For all ⅋-links L of $\varphi_{\mathcal{S}}(\Theta)$, we draw an edge between the premise (left or right) selected by $\mathcal{S}(L)$ and the conclusion.*

Definition 19

A proof-structure Θ is said to be a proof-net *when for all switchings $\mathcal{S}$, the graph $\Theta_{\mathcal{S}}$ is connected and acyclic.*

We immediately get the

Theorem 3

If Θ is sequentializable, then Θ is a proof-net.

Proof. — The proof is straightforward and uninteresting. □

E.4 The sequentialization theorem

Theorem 4

Proof-nets are sequentializable.

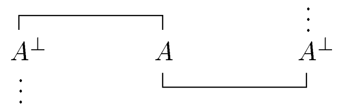
This is a non-trivial result, first proved in [12] ; what we have indeed presented here is a simplified version of the criterion, due to Danos and Regnier, [6].

E.5 Cut-elimination for proof-nets

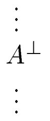
The crucial test for the new syntax is the possibility to handle syntactical manipulations directly at the level of proof-nets (therefore completely ignoring sequent calculus). When we meet a cut link

we look at links whose conclusions are A and $A^{\perp}$:

(1) One of these links is an axiom link, typically :



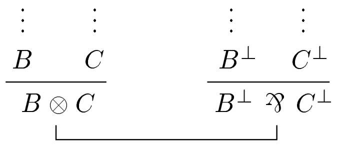
such a configuration can be replaced by



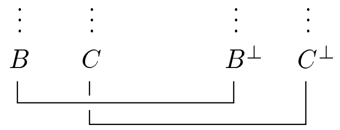
however the graphism is misleading, since it cannot be excluded that the two occurrences of $A^\perp$ in the original net are the same ! But this would correspond to a configuration

and such configurations are excluded by the correctness criterion.

(2) If both formulas are conclusions of logical links for $\otimes$ and $\invamp$, typically



then we can replace it by

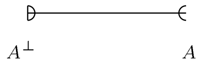


and it is a very interesting exercise to show that the new structure still enjoys the correctness criterion. This cut-elimination procedure has very nice features :

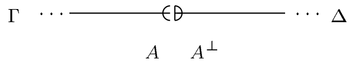
1. It enjoys a Church-Rosser property, i.e. it is deterministic
2. It is linear in time : simply observe that the proof-net shrinks with any application of steps (1) and (2) ; this linearity is the start of a line of applications to computational complexity, typically [16].
3. The treatment of the multiplicative fragment is purely local ; in fact all cut-links can be simultaneously eliminated. This must have something to do with parallelism and by the way Yves Lafont developed his *interaction nets* as a kind of parallel machine working like proof-nets [21, 23].

E.6 Geometry of interaction

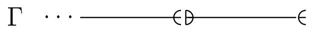
By far the best explanation of the identity links can be taken from *electronics*. Think of a sequent Γ as the interface of some electronic equipment, this interface being made of plugs of various forms $A_1, \ldots, A_n$; the negation corresponds to the complementarity between male and female plugs. Now a proof of Γ can be seen as any equipment with interface Γ. For instance the axiom link is such a unit and it exists in everyday life as the *extension cord* :



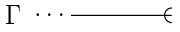
Now, the cut link is well explained as a plugging :



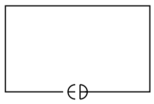
The main property of the extension cord is that



behaves like



It seems that the ultimate, deep meaning of cut-elimination is located there. Moreover observe that common sense would forbid self-plugging of an extension cord :



which would correspond, in terms of proof-nets to the incestuous configuration :

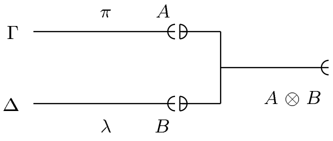
which is not acknowledged as a proof-net ; in fact in some sense the ultimate meaning of the *correctness* criterion (theorem 4) is to forbid such a configuration (and also disconnected ones).

This analogy with plugs is the starting point of *Geometry of interaction.* Can we push our electronic analogy so as to accommodate the other links ?

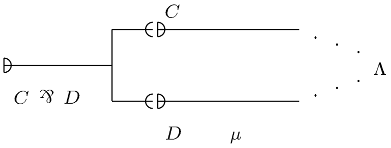
Let us first precise the nature of our (imaginary) plugs ; there are usually several pins in a plug. We shall restrict ourselves to one-pin plugs ; this does not contradict the fact that there may be a huge variety of plugs, and that the only allowed plugging is between complementary ones, labelled A and $A^\perp$.

The interpretation of the rules for $\otimes$ and $⅋$ both use the following well-known fact : two pins can be reduced to one (typical example : stereophonic broadcast), which is called *multiplexing.*

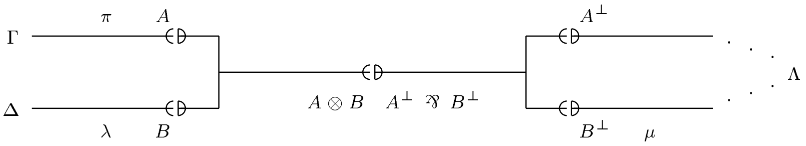
- $\otimes$-*rule :* from units π, λ with respective interfaces $\vdash \Gamma, A$ and $\vdash \Delta, B$, we can built a new one by merging plugs A and B into another one (labelled $A \otimes B$) by means of a multiplexer.



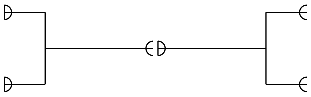
- $\parr$ -*rule :* from a unit μ with an interface $\vdash C, D, \Lambda$, we can built a new one by merging plugs C and D into a new one (labelled $C \parr D$) by means of a multiplexer :



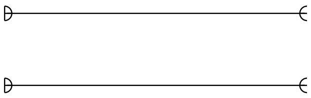
To understand what happens, let us assume that $C = A^\perp$, $D = B^\perp$; then $A^\perp \parr B^\perp = (A \otimes B)^\perp$, so there is the possibility of plugging. We therefore obtain



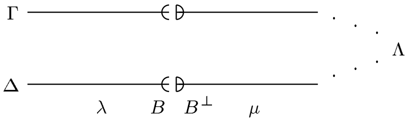
But the configuration



is equivalent to (if the muliplexers are the same)



and therefore our plugging can be mimicked by two pluggings



Moreover, if we remember that coding is based on a development by means of Fourier series (which involves the Hilbert space) everything that was done can be formulated in terms of operator algebras : multiplexing can be interpreted by means of an isometry $x \oplus y \rightsquigarrow p(x) + q(y)$ of $\mathbb{H} \oplus \mathbb{H}$ into $\mathbb{H}$, equivalently by means of two partial isometries $p^*p = q^*q = 1, p^*q = q^*p = 0$. In fact the operator algebra semantics enables us to go beyond multiplicatives and quantifiers, since the interpretation also works for exponentials. We shall not go into this, which requires at least some elementary background in functional analysis ; however, we can hardly resist mentioning the formula for cut-elimination

$$\mathrm{EX}(u, \sigma) := (1 - \sigma^2)u(1 - \sigma u)^{-1}(1 - \sigma^2)$$

which gives the interpretation of the elimination of cuts (represented by σ) in a proof represented by u. Termination of the process is interpreted as the nilpotency of σu, and the part $u(1 - \sigma u)^{-1}$ is a candidate for the execution. See [14], for more details.
Let us end this subsection with yet another refutation of weakening and contraction :

1. If we have a unit with interface $\vdash \Gamma$, it would be wrong to add another plug A ; such a plug (since we know nothing about the inside of the unit) must be a mock plug, with no actual connection with the unit ... Imagine a plug on which it is written “danger, 220V”, you expect to get some result if you plug something with it : here nothing will happen !

2. If we have a unit with a repetitive interface $\vdash \Gamma, A, A$, it would be wrong to merge them into a single one : in real life, we have such a situation with the stereophonic output plugs of an amplifier, which have exactly the same specification. There is no way to merge these two plugs into one and still respect the specification. More precisely, one can try to plug a single loudspeaker to the two outputs plugs simultaneously ; maybe it works, maybe it explodes, but anyway the behavior of such an experimental plugging is not covered by the guarantee ...

E.7 Proof-nets and duality

The correctness criterion, i.e. theorem 4 can indeed be seen as the closest prefiguration of our program. So assuming that we have a proof-structure Θ, i.e. a would-be proof of a single formula A, without cut, and assuming that we are given a switching $\mathcal{S}$ of Θ, we shall produce a paraproof of $\vdash A^\perp$. As usual, we start with the conclusion, until we reach axioms. The formulas occurring in the sequents of our paraproof will be the negations of the formulas of Θ. The (non)-deterministic algorithm is as follows (sequents are considered up to order, i.e. modulo exchange) :

1. If I get a sequent $\vdash \Gamma, B \parr C$, and if $B^\perp, C^\perp$ occur in Θ, then I can apply a $\parr$-rule, with the sequent $\vdash \Gamma, B, C$ as premise.

2. If I get a sequent $\vdash \Gamma, B \otimes C$, and if $B^\perp, C^\perp$ occur in Θ as the premises of a $\parr$-link L, then I can apply a $\otimes$-rule whose premises are
   - $\vdash \Gamma, B$ and $\vdash C$ if $\mathcal{S}(L) = l$
   - $\vdash B$ and $\vdash \Gamma, C$ if $\mathcal{S}(L) = r$

3. Otherwise $\vdash \Gamma$ is accepted as an axiom (paralogism « Give up »)

Such a paraproof can be represented as a proof-structure, which is exactly as a usual one, but for the fact that arbitrary axioms (i.e. links with no premises and the formulas of Γ as

conclusions) are used to represent the axiom $\vdash \Gamma$; except for this detail, this proof-structure is indeed a (para-) proof-net, which is indeed uniquely determined by $\mathcal{S}$.
Now, remember our concern

1. Cut-elimination still holds.

2. Paralogisms produce « enough » paraproofs.

As to (1), I can perform a cut between my proof-net of A and my paraproof-net of $A^{\perp}$ and perform cut-elimination in this paraproof-net, up to the moment where all $\otimes$ and $⅋$-links have been eliminated. Geometrically I end with a connected and acyclic structure, containing only axioms and cuts. This corresponds to the necessity of the criterion, theorem 3.
As to (2), if I consider those paraproof-nets coming from switchings, then the sufficiency of the criterion, theorem 4 enables one to sequentialize my proof-net. Indeed the switchings should be seen as a *dense* subset of paraproofs.
The homogeneity between proofs and paraproofs is total and proofs are distinguished by the fact of using legal axioms (this can be given a geometrical explanation too, but I hope that I did enough to illustrate the idea).
This interpretation is asynchronous, i.e. it is enough to consider the *result* of the cut-elimination, which is a connected and acyclic graph. Everything can be reformulated in terms of partitions and duality of partitions, see [6].

E.8 Non-commutativity

If we draw a proof-net on a sheet, we shall observe crossings between axiom links. This is due to the rule of exchange, which expresses commutativity. But if we forbid exchange (indeed we can accept circular exchange), then the proofs become *planar*. Abrusci has recently found an alternative criterion which does not mention planarity. This criterion is homogeneous with my original criterion for proof-nets, and this is why it works simultaneously for commutative and non-commutative logics, which are therefore compatible, [3]. My original criterion was formulated in terms of permutations (indeed partitions are forgetfully obtained as the cycles of the permutations). This indicates where to look in order to be non-commutative.
... But partitions or permutations or operators, etc. will never make it, since we cannot abolish time !

F Time

Here we shall be concerned with logical time. Something must be clear to start with, namely that time cannot be located in the bleak bureaucracy known as temporal « logics » : time is about dynamics, not kinematics... But it is more difficult to find its precise location. We shall below interpret time as a failure of association, i.e. as a mismatch between connectives. In fact the crucial point is the proof of associativity of the Tensor :

F.1 Associativity

We first prove a lemma :

Proposition 3

If X, Y are arbitrary subsets of M, then $(X.Y^{\perp\perp}) \subset (X.Y)^{\perp\perp}$.

Proof. — Let $x \in X, b \in Y^{\perp\perp}, z \in (X.Y)^{\perp}$: we want to show that $xbz \in \perp$. For this take $y \in Y$ and observe that $zxy \in \perp$, which shows that $zx \in Y^{\perp}$, from which $bzx \in \perp$. □
Associativity easily follows : $A \otimes (B \otimes C) = (A.(B.C)^{\perp\perp})^{\perp\perp} = (A.(B.C))^{\perp\perp} = (A.B).C))^{\perp\perp} = ((A.B)^{\perp\perp}.C)^{\perp\perp} = (A \otimes B) \otimes C$.
By the way, our argument is still valid in the absence of commutativity, provided $\perp$ is cyclic, i.e. $xy \in \perp \Rightarrow yx \in \perp$. In general, one can define two orthogonals, $X^{\perp} = \{x; xX \subset \perp\}$ and $X^{\top} = \{x; Xx \subset \perp\}$, and cyclicity equals them. In the absence of cyclicity, we get left and right facts, right facts being those X of the form $Y^{\top}$, i.e. $X = X^{\perp\top}$. If we define the tensor as $(X.Y)^{\perp\top}$, then we cannot prove our lemma any longer, i.e. associativity fails : the same proof works, but we cannot finally replace bzx with xbz.

F.2 Positivity

Imagine now that we want to prove that $\otimes$ distributes over $\oplus$, a crucial property ; we shall prove a similar lemma for $\oplus$:

$$(X \cup Y^{\perp\perp}) \subset (X \cup Y)^{\perp\perp}$$

from which associativity can be reduced to $X.(Y \cup Z) = X.Y \cup X.Z$.
Say that a monotonous function Φ from $\wp(M)$ to itself is ***positive*** when it enjoys $\Phi(X^{\perp\perp}) \subset \Phi(X)^{\perp\perp}$. Then union and product are binary positive functions. By the way all 0-ary functions (i.e. all sets) enjoy the $0-ary$ analogue, $X \subset X^{\perp\perp}$, in particular $\emptyset$ and $\{1\}$. The property of positive operations, is that when performing a cluster of such operations, we do not need any biorthogonal, but the final one, typically instead of $\Psi(\Phi(X^{\perp\perp})^{\perp\perp})^{\perp\perp}$, we can content ourselves with the simpler $\Psi(\Phi(X))^{\perp\perp}$. This is the technical meaning of the associativity (in the general sense of ***association***) of positive operations. In terms of time, a step is definitely performed when we alternate ***polarities***, i.e. perform a negative operation after a positive one. This is reflected in our approach (the hypersequentialized calculus) in which the basic rules are clusters of rules of the same polarity.

F.3 Negativity

A ***negative*** function is just a monotonous function mapping facts to facts. So are $\& = \cap$ and $\parr$, and so are the constants $\perp$ and $\top = M$. Negative operations associate too, the only difference being that the biorthogonals must be performed first. Now if we iteratively apply positive and negative operations, the only moment when we really need the biorthogonal is when a positive operation Ψ is followed by a negative one Φ : $\ldots \Phi(\Psi(\ldots)^{\perp\perp}) \ldots$. The need for double orthogonal causes the failure of associativity between the two groups, typically, in spite of the distributivities between $\cap$ and $\cup$, there is no distributivity between $\&$ and $\oplus$: $X \cap (Y \cup Z)^{\perp\perp}$ cannot be simplified.

F.4 Association

Jean-Marc Andreoli [4] was the first person to stress this distinction. In terms of proof-search, the connectives $\perp, \top, \&, \parr$ are ***invertible*** : this means that when we meet a formula starting with one of these connectives in a sequent, we can decide that the last rule is THE (unique) rule for this connective, e.g. if we meet $\vdash \Gamma, A ⅋ B$, we can decide that the last rule is the ⅋-rule with the premise $\vdash \Gamma, A, B$. « We can decide » means that, whatever we do, we are bound to use this rule later, and that this will change nothing [37]. Andreoli called these connectives ***asynchronous***, because of their indifference to time. Observe that negative operations associate, simply make a cluster of all negative operations iteratively available. Given any sequent, we can deterministically reduce its provability (indeed its proofs) to the provability of other sequents made of positive formulas.

In this way, we can easily justify the associativity of the ⅋, or its distributivity over & : the sequents to which one can reduce $\vdash \Gamma, A ⅋ (B \& C)$ and $\vdash \Gamma, (A ⅋ B) \& (A ⅋ C)$ are the same, namely $\vdash \Gamma, A, B$ and $\vdash \Gamma, A, C$.

By the way, we established (through duality) the distributivity of $\otimes$ over $\oplus$, by invertibility, and there must be a dual property. The positive connectives are far from being invertible : there is no rule for **O**, the rule for **1** only works in presence of an empty context, and the rules for $\otimes$ and $\oplus$, seen from below, i.e. from the viewpoint of proof-search, are very problematic

1. If we decide to apply the $\otimes$-rule to get the conclusion $\vdash \Gamma, A \otimes B$, then we have to guess a splitting of Γ into Γ', Γ", Γ' going with A, and Γ" going with B.

2. If we decide to apply the $\oplus$-rule to get the conclusion $\vdash \Gamma, A \oplus B$, then we have to decide between A and B.

3. The problem is not only that when deciding to apply a rule we are forced to make choices : deciding to apply a rule at all is already a commitment : given $\vdash A \otimes B, C \oplus D$, we must choose between the $\otimes$ and the $\oplus$, since in general only one of them can be performed.

Andreoli's observation, known as *focalization*, does not fix drawbacks (1) and (2), but drastically reduces (3) : if we choose the right moment to use a $\otimes$-rule, then we can immediately perform a cluster of positive rules, up to the moment we reach negative constituents. Typically, take $\vdash \Gamma, A \otimes (B \otimes C)$; I am not sure that the last rule is a $\otimes$-rule, but if I accept to bet, I can immediately proceed with the internal tensor, i.e. it is not more dangerous to split into A and B and C than into A and $B \otimes C$.

This is indeed the algorithmic contents of our proof of associativity : if A is interpreted as the set of contexts Γ such that $\vdash \Gamma, A$ is provable, then $A^\bullet.B^\bullet$ is the set of contexts Γ such that $\vdash \Gamma, A \otimes B$ is provable, with the $\otimes$-rule as the last rule, and the double orthogonal corresponds to the general case when the last rule might work on the context Γ ; the abuse of biorthogonals corresponds to an alternation between the context and $A \otimes B$, and positivity enables one to make only one final alternation.

The distinction positive/negative makes sense for quantifiers as well : $\forall$ is negative, $\exists$ is positive. As to exponentials, the situation is more complex : « ! » behaves like a combination $\Phi\Psi$, Φ positive, Ψ negative, and therefore exponentials have a strange associative behavior, reflected in $!(A \& B) = !A \otimes !B$, the only socialization between the negative & and the positive $\otimes$. Exponentials are therefore strongly temporal ; they definitely scanse (at least) one step.

F.5 Synthetic connectives

A current misconception, which has been mine for a long time, is to consider that any formula $\phi[A_1, \ldots, A_n]$ defines a connective. This is indeed not the case, since we may fail

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[37] There is something like this, in the theory of the Chess openings, where one can reach the same position through transpositions of moves : this has a tactical value, namely to reduce possibilities of the opponent. The same occurs here : if we perform the invertible operations first, we augment the possibilities, i.e. invertible rules are the locks which command non-invertible ones.

to find specific rules for this connective (and the rules for its dual $\phi[A_1{}^{\perp}, \ldots, A_n{}^{\perp}]^{\perp}$). The condition for the existence of specific rules is exactly that we can perform the deconstruction of ϕ as a cluster. This is exactly true when ϕ is homogeneous, i.e. is a cluster of connectives of the same polarity. Let us give two examples :

1. $P \oplus (Q \otimes R)$ defines a ternary connective, with two rules, which can be justified by focalization ; the rule of the dual $P \,\&\, (Q \parr R)$ is obtained by invertibility.
2. $P \oplus (Q \,\&\, R)$ does not define a connective ; in fact its dual $P \,\&\, (Q \oplus R)$ is problematic : the two rules « From $\vdash \Gamma, P$ and $\vdash \Gamma, Q$ derive $\vdash \Gamma, P \,\&\, (Q \oplus R)$ » and « From $\vdash \Gamma, P$ and $\vdash \Gamma, R$ derive $\vdash \Gamma, P \,\&\, (Q \oplus R)$ » are not enough. For instance they will not be sufficient to prove $\vdash P^{\perp} \oplus (R^{\perp} \,\&\, Q^{\perp}), P \,\&\, (Q \oplus R)$.

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