On the meaning of logical rules II : multiplicatives and additives

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Abstract

El método inicial que imagino era relativamente sencillo. Conocer bien el español, recuperar la fe católica, guerrear contra los moros o contro el turco, olvidar la historia de Europa entre los años de 1602 y de 1918, *ser* Miguel de Cervantes. Pierre Ménard estudió ese procedimiento (sé que logró un manejo bastante fiel del español del siglo diecisiete), pero lo descartó por fácil. [...]

Mi complaciente precursor no rehusó la colaboración del azar : iba componiendo la obra inmortal un poco *à la diable*, llevado por inercias del lenguaje y de la invención. Yo he contraí el misterioso deber de reconstruir literalmente su obra espontánea. Mi solitario juego está gobernado por dos leyes polares. La primera me permite ensayar variantes de tipo formal o psicológico ; la secunda me obliga a sacrificarlas al texto « original » y a razonar de un modo irrefutabile esa aniquilación.

J.-L. Borges *Pierre Ménard autor del Quijote*, 1939.

The paper expounds the solution to our search for meaning [3] in a particular case : the fragment of logic built from the neutral elements $\perp, \top, \mathbf{1}, \mathbf{0}$ by means of the connectives $\parr, \&, \otimes, \oplus$.

As explained in the previous paper and summarized in the abstract, the task is to produce a trivial meet between syntax and semantics, but in a non-trivial way. Typically, we cannot content ourselves with a plain game-theoretic paraphrases of logic ; we have to deconstruct familiar syntax and to reconstruct it in another way. But, as in the case of Ménard reconstructing the *Quijote*, the synthesis is absolutely prior to the analysis... in particular the presentation analysis/synthesis that we follow is completely alien to the very spirit of the work.

We shall organize a sort of tunnel between syntax and semantics, with a meet in between : the analysis will replace syntax with a game-theoretic variant, and synthesis will elaborate an abstract notion of game, which can be specialized to our fragment and yield exactly the analytic games.

Analysis We shall modify the extant sequent calculus into a *hypersequentialized* version **HC**, which has internalized the alternation positive/negative to the extent that even negative formulas disappear... the usual connectives being replaced with *synthetic connectives*. The configurations of **HC** can be fully analyzed by means of various game-theoretic artefacts. Moreover, **HC** admits wrong logical rules (*paralogisms* [1]), which are essential to prove the basic equivalences between usual logic and the game version in **HC**. These paralogisms have a heavy price : who uses them loses, and we end with a complete equivalence between usual logic and *winning strategies* in a sort of analytic game.

Synthesis We can, starting with the geometrical idea of iterated division of space, build a universal game. The *disputes* (i.e. the plays) of this game are equipped with a (sort of) structure of coherent space. Then cliques in this coherent space may be seen as *designs* (i.e. a very specific sort of strategy), and when a design is *orthogonal* to a counterdesign, the unique dispute they share has at most one winner. If we define a *behavior* as a set of designs equal to its biorthogonal, then logical formulas are interpreted as behaviors. Then it is a matter of care to translate a design within a logical behavior into a paraproof of the corresponding formula, and a winning design into a real proof. By the way, losing corresponds to giving up or making a deliberate mistake to annoy the enemy, what is called here a *dog's move*, whose basic meaning (besides the fact that it might also make your opponent lose) is that of an artificial obstruction : this is the geometrical meaning of paralogisms.

Eventually, the paper ends into a trivial equivalence, what was sought... But this quest for triviality was highly difficult : if the rest of the connectives is not yet available, there is a reason for that : there is still a lot to do before reaching a state of highly non-trivial triviality.
By the way, the framework here is only an approximation to the complete solution : it seems necessary to present the material in a progressive way, first a simplified case (in which for instance $8bf1$ is not quite neutral), and then slightly complicate the pattern, the main avenues remaining unchanged.

Remark. — Our interpretation is based on seven pillars, that we distinguish by different fonts/styles, namely :

- Fingers (in French : doigts ; intuition : subformula index) $i, j, k, \ldots$
- Hands (in French : mains ; intuition : formula) $\sigma, \xi, \eta, \ldots$
- Blokes (in French : gonzes ; intuition : sequent) $\Xi \vdash \Sigma$, or $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$
- Mauls (in French : mêlées ; intuition : proof of the empty sequent) $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \ldots$
- Disputes (in French : disputes ; intuition : process of normalization) $[\mathfrak{m}_{\mathfrak{o}}, \ldots, \mathfrak{m}_{\mathfrak{n}}]$ or $\mathfrak{D}, \mathfrak{E}, \mathfrak{F}, \ldots$
- Designs (in French : desseins ; intuition : proofs and paraproofs) $\mathcal{S}, \mathcal{T}, \mathcal{R}, \ldots$
- Behaviors (in French : comportements ; intuition : the meaning of a formula) $\mathbf{G}, \mathbf{H}, \mathbf{K}, \ldots$

1 The hypersequentialized calculus

We recall that formulas are classified according to their *polarity* :

- $\mathbf{1}, \mathbf{0}$ and formulas starting with $\otimes, \oplus$ are *positive.*
- $\bot, \top$ and formulas starting with $⅋, \&$ are *negative.*

We shall later introduce the atom $\mathbf{0}^\flat$ (positive), the *boot.* As usual we should pay attention to repetitions of formulas, i.e. the phenomenon of *occurrences*. But there is a very radical way to do so, namely to have infinitely many constants —say $\mathbf{0}_i^\flat, \mathbf{0}_i$— and to use systematically fresh constants.

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[1] To avoid misunderstandings : the mistakes are *volontary*, nothing to do with —say— « abduction ».

1.1 Synthetic connectives

Take a formula, say positive ; then we can decompose it into its subformulas, and go on until we stop, either because we reach an atom like $\mathbf{0}$, or a negative formula. Then our formula looks like $P = \psi[N_1, \ldots, N_k]$, where ϕ is a *cluster* of positive connectives and the N_i are negative. We can rewrite this as $P = \phi[P_1, \ldots, P_k]$, where the P_i are the negations of the N_i. Typically $L \oplus (M \otimes N)$ will appear as a ternary operation $\phi[P, Q, R] = P^\perp \oplus (Q^\perp \otimes R^\perp)$. The major property of these new connectives is that they lead from positive to positive, i.e. that there is no longer any negative formula. This is made technically possible by the *focalization property* of Andreoli [1], which states the possibility of handling a cluster of connectives of the same polarity as a single connective, see also [3], annex F.

Proposition 1

If X is a finite coherent space, then the following are equivalent :

- *X is series/parallel, i.e. it can be built from the coherent space $\mathbf{1}$ by means of the constructions & (series) and $\oplus$ (parallel).*
- *X harbors no* horseshoe, *i.e. no 4-element subgraph such that $a \frown b \frown c \frown d$ and $b \smile d \smile a \smile c$.*
- *X can be contracted into one point by iteration of the operation : if x, y are undiscernible, i.e. $\forall z \neq x, y \quad (z \frown x \Leftrightarrow z \frown y)$, then remove one of x, y.*

Such an X is called contractile. *If $X \neq \emptyset$ and C is a maximal clique in X and D is a maximal anticlique in X, then $C \cap D \neq \emptyset$.*

PROOF. — Folkloric. □

Definition 1

A n-ary synthetic connective *$\mathfrak{X}$ is a finite set of propositional variables $\{p_1, \ldots, p_n\}$ equipped with a structure of contractile coherent space.*

Typically :

1. The empty coherent space (corresponding to the constant $\mathbf{0}$ of linear logic)
2. The binary connective $p_1 \frown p_2$, which corresponds to the connective ${p_1}^\perp \otimes {p_2}^\perp$ of linear logic.
3. The binary connective $p_1 \smile p_2$, which corresponds to the connective ${p_1}^\perp \oplus {p_2}^\perp$ of linear logic.
4. There is no synthetic connective corresponding to the constant $\mathbf{1}$; in fact the only reasonable candidate would be the empty set which already takes care of the constant $\mathbf{0}$; a coherent space with one point corresponds to the positivization of negation, i.e. $\downarrow p$ is $p^\perp$ made positive [2]. The constant $\mathbf{1}$ will therefore be $\downarrow \mathbf{0}^\flat$, where $\mathbf{0}^\flat$ is a new constant specific of **HC**. By the way, interpreting $\mathbf{1}$ as $\emptyset$ would force $\mathbf{1} \oplus \mathbf{1}$ to be empty as well, but this would cause a degeneracy of the interpretation, since $\mathbf{1} \oplus \mathbf{1}$ is a reasonable interpretation of booleans.

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[2] In terms of games, a positive formula corresponds to « I start », and negation is the interchange between players ; $\downarrow p$ is still positive, i.e. a dummy first move being added : $\downarrow$ is not involutive, since doing it twice will produce the same plays, but with two additional dummy moves, which do matter !

Definition 2

Let $\phi[N_1,\ldots,N_k,\mathbf{1}_1,\ldots,\mathbf{1}_l]$ be a positive formula built on the formulas $N_1,\ldots,N_k$ (negative) and $\mathbf{1}_1,\ldots,\mathbf{1}_l$ (occurrences of $\mathbf{1}$), by means of $\mathbf{0}$, $\oplus$, $\otimes$. Let $\mathbf{0}^\flat_1,\ldots,\mathbf{0}^\flat_l$ be occurrences of the (positive) atom $\mathbf{0}^\flat$. Then we construct a synthetic connective $\mathfrak{X}$ on a subset of the positive atoms ${N_1}^\perp,\ldots,{N_k}^\perp,\mathbf{0}^\flat_1,\ldots,\mathbf{0}^\flat_l$:

1. *If $\phi=\mathbf{0}$, then $\mathfrak{X}$ is the empty coherent space.*
2. *If $\phi=\mathbf{1}_i$ (resp. N_i) then $\mathfrak{X}$ is the unit coherent space on the atom $\mathbf{0}^\flat_i$ (resp. ${N_i}^\perp$).*
3. *If $\phi=\phi_1\oplus\phi_2$, then $\mathfrak{X}=\mathfrak{X}_1\oplus\mathfrak{X}_2$ (i.e. the parallel sum of the coherent spaces $\mathfrak{X}_i$)*
4. *If $\phi=\phi_1\otimes\phi_2$, then $\mathfrak{X}=\mathfrak{X}_1\ \&\ \mathfrak{X}_2$ (i.e. the series sum of the coherent spaces $\mathfrak{X}_i$), except if one of the $\mathfrak{X}_i$ is empty, in which case $\mathfrak{X}$ is empty as well.*

Proposition 2

Let ϕ, $\mathfrak{X}$ be as above, and let X be a non-empty maximal clique in $\mathfrak{X}$; then

1. *$\mathfrak{X}$ cannot be empty, i.e. if $\phi=\mathbf{0}$, there is no such X.*
2. *If ϕ is $\mathbf{1}_i$ or N_i, then X is the unique singleton clique of $\mathfrak{X}$.*
3. *If ϕ is $\phi_1\oplus\phi_2$, then X is either a maximal non-empty clique in $\mathfrak{X}_1$ or (exclusive or) in $\mathfrak{X}_2$.*
4. *If ϕ is $\phi_1\otimes\phi_2$, then X is the union of a non empty maximal clique in $\mathfrak{X}_1$ and a non-empty maximal clique in $\mathfrak{X}_2$: this excludes the case where one of $\mathfrak{X}_1,\mathfrak{X}_2$ is empty.*

PROOF. — Obvious ; the proposition explains the peculiarities of definition 2. $\square$

1.2 The syntax of HC

Definition 3

The formulas in **HC** *are defined inductively by :*

1. *The atom $\mathbf{0}^\flat$ is a formula.*
2. *If $\mathfrak{X}$ is a synthetic connective on the variables $p_1,\ldots,p_n$ and if $P_1,\ldots,P_n$ are formulas, then $\mathfrak{X}P_1,\ldots,P_n$ is a formula.*

From what precedes, one can associate to any positive formula P written in the fragment $\mathbf{1},\mathbf{0},\bot,\top,\otimes,\oplus,\parr,\&$, a unique formula of **HC** : one writes P as $\phi[N_1,\ldots,N_k,1_1,1_l]$, where ϕ is purely positive and the N_i are negative, and one replaces ϕ with the appropriate $\mathfrak{X}$, applied to some of the ${N_i}^\perp,\mathbf{0}^\flat_j$; the $\mathbf{0}^\flat_j$ are just occurrences of $\mathbf{0}^\flat$ whereas the ${N_i}^\perp$ can in turn be analyzed in the same way. Observe that $\mathbf{0}^\flat$ does not correspond to anything in linear logic, in particular one should not imagine that $\mathbf{0}^\flat$ is a missing constant of LL (expressions like $\mathbf{0}^\flat\oplus P$ do not seem to make sense in LL).

1.3 The calculus HC : cut-free part

Definition 4

A sequent *$\Xi\vdash\Sigma$ consists in two multisets of formulas Ξ,Σ ; there must be at least one formula in $\Xi\cup\Sigma$ and at most one formula in Ξ. A sequent with Ξ empty is said to be* major, *a sequent $\xi\vdash\Sigma$ is said to be* minor. *We shall consider that we are working with sets, i.e. that we are indeed dealing with occurrences.*

Definition 5

The rules of the (cut-free) sequent calculus **HC** *are defined below ; some of them will be considered as* paralogisms, *and therefore the other rules will be considered as* correct.

1. *The hypothesis* $\vdash \Sigma$ *for any major sequent (paralogism) ; the expression* hypothesis *means that the proof-tree is not quite well-founded, i.e. that something is missing.*
2. *The axiom* $\mathbf{0}^\flat \vdash \Sigma$ *(paralogism, unless* Σ *only consists in occurrences of* $\mathbf{0}^\flat$*).*
3. *Let* $\mathfrak{X}P_1, \ldots, P_n$ *be a formula, and let* X *be a non-empty maximal clique of* $\mathfrak{X}$*, e.g.* $X = \{p_{i_1}, \ldots, p_{i_k}\}$*. Then we are allowed to use the following rule : from* $P_{i_1} \vdash \Sigma_1, \ldots, P_{i_k} \vdash \Sigma_k$ *deduce* $\vdash \mathfrak{X}P_1, \ldots, P_n, \Sigma_1, \ldots, \Sigma_k$*.*
4. *Let* $\mathfrak{X}P_1, \ldots, P_n$ *be a formula ; for each non-empty maximal clique* X *of* $\mathfrak{X}$*, e.g.* $X = \{p_{i_1}, \ldots, p_{i_k}\}$*, let us introduce* $\Upsilon_X = P_{i_1}, \ldots, P_{i_k}$*. Then we are allowed to use the following rule : from* $\vdash \Upsilon_X, \Sigma$ *for each* X*, deduce* $\mathfrak{X}P_1, \ldots, P_n \vdash \Sigma$*.*

A paraproof *is anything generated by these rules ; a* proof *is a paraproof without paralogism.*

Our distinction between axiom and hypothesis comes from the fact that in 1. Σ is major, and that a rule introducing a major sequent should have at least one premise. This is not the case for minor sequents, which have a well-defined rule introducing them, which may be without premise ; this is why 2. is called axiom. By the way, the case $\mathfrak{X} = \emptyset$ (i.e. $\mathbf{0}$) produces an axiom too, see below. It is quite easy to see that :

1. When $\mathfrak{X}$ is empty, then there is no rule introducing $\mathfrak{X}$ to the right, but $\mathfrak{X} \vdash \Sigma$ is an axiom : this corresponds to the connective $\mathbf{0}$.
2. The rules of $\mathfrak{X}[P_1, P_2]$ will correspond to the rules for the tensor ${P_1}^\perp \otimes {P_2}^\perp$ (if $P_1 \frown P_2$) or the sum ${P_1}^\perp \oplus {P_2}^\perp$ (if $P_1 \smile P_2$).
3. The constant $\mathbf{1}$ is represented by $\downarrow\mathbf{0}^\flat$, but the situation is more complex :
    - (a) Since the only correct axioms are of the form $\mathbf{0}^\flat \vdash \mathbf{0}^\flat, \ldots, \mathbf{0}^\flat$, a real proof of a sequent involving $\downarrow\mathbf{0}^\flat$ to the right must somewhere use the right rule for $\downarrow$, i.e. derive some $\vdash \Sigma, \downarrow\mathbf{0}^\flat$ from $\mathbf{0}^\flat \vdash \Sigma$, whose only real proof consists in an axiom, but Σ must then consist of occurrences of $\mathbf{0}^\flat$. This corresponds to the « derived axiom » $\vdash \mathbf{1}, \mathbf{0}^\flat, \ldots, \mathbf{0}^\flat$.
    - (b) Now given any paraproof of $\Xi \vdash \Sigma$, one can produce a « weakened » paraproof of $\Xi \vdash \Sigma, \mathbf{0}^\flat$, by adding some $\mathbf{0}^\flat$ to the right of all sequents of the paraproof. Moreover, this operation preserves correctness. This indeed justifies the derived rule « From $\vdash \Sigma$ derive $1 \vdash \Sigma$ », which is weakening.
    - (c) We therefore see that the rules for $\mathbf{1}$ and $\perp$ are faithfully represented. However the case of weakening is awfully non deterministic : typically, in the case of $\vdash \mathfrak{X}PQ$, where $\mathfrak{X}$ means $p^\perp \otimes q^\perp$, the addition of $\mathbf{0}^\flat$ to get $\vdash \mathfrak{X}PQ, \mathbf{0}^\flat$ will involve a choice between P and Q, e.g. to modify the premises $P \vdash$ and $Q \vdash$ into $P \vdash \mathbf{0}^\flat$ and $Q \vdash$. In other terms, we rediscover the absence of satisfactory proof-nets for the multiplicative neutrals, that we shall formulate in a more drastic way : the multiplicative neutrals are not quite neutral, at least in the present setting. In the next paper [4], they will be neutral. But we refuse to introduce the full setting corresponding to exponentials for such a marginal issue.

From this it is quite easy to prove a correspondence between (real) cut-free proofs in **HC** and cut-free proofs in the fragment of linear logic under study. Let us for instance look at the less trivial part, i.e. how we can translate a proof of a sequent $\vdash A$ in our fragment into a **HC**-proof.

1. We write the final sequent as $\vdash A$ if A is positive, or $A^{\perp} \vdash$ if A is negative. We also modify the proof so as to use no identity axiom : « reverse η-conversion ».
2. We are iteratively led to the more general situation of a cut-free proof of a sequent $\Xi \vdash \Sigma$, made of positive formulas, and with at most one formula in Ξ. Moreover the proof does not use the identity axiom of linear logic.
3. If $\Xi = P$, then, by appropriate permutations of rules, we can make sure that the last rule of the proof introduces P to the left, typically $P = A \otimes B$, then the premise is $A, B \vdash \Sigma$, and in case one of A, B is not negative, we iterate the process. Eventually we get a portion of proof using only left rules, and starting with sequents $\Theta_i \vdash \Sigma$, where the Θ_i are non-empty multisets of negative formulas. Indeed I can put the Θ_i to the right by negating them, and I can produce a bijection $f : {\Theta_{f(X)}}^{\perp} = \Upsilon_X$ so that the sequents $\vdash \Sigma, \Upsilon_X$, are exactly the premises of the left rule of the appropriate synthetic connective.
4. If $\Xi = \emptyset$, then we can apply focalization, i.e. that —up to permutation of rules— one of the formulas of $\vdash \Sigma$, say P, is such that a cluster of right rules has been applied, up to the negative constituants of P ; the premises of the cluster are of the form $\vdash \Sigma_i, N_i$, and if we replace them by ${N_i}^{\perp} \vdash \Sigma_i$ we are in the position to apply one of the right rules of the appropriate synthetic connective. Observe that, due to the possibility of alternative focalizations, several distinct **HC**-proofs can arise in this way.
5. One should be slightly more precise with $\mathbf{1}, \perp$, but this is a minor pedantic issue.
6. Last but not least : no paralogism is used in the translation.

1.4 The full HC

The consideration of cut forces one to use some proof-net technology, i.e. boxes and cut-links, see [3].

Definition 6

By a box *we mean a sequent, together with a* justification. *The justification can be —at this stage— rather arbitrary, i.e. boxes interact through the formulas of the sequent, which are the* doors *of the box ; observe that a box has at most one* main door, *i.e. the formula which is on the left-hand side of the sequent, the others being* auxiliary *; the main door P is indicated by $P \vdash$ —after all the turnstile is a sort of door. Given boxes $\mathcal{B}_0, \ldots, \mathcal{B}_n$, we can produce a* cut-net, *by means of only two links :*

- *The boxes, seen as generalized axioms (conclusions : the formulas of the sequent, but the main door is understood « negatively »).*
- *The cut-link, which links a main door with an auxiliary door, which are occurrences of the same formula. The cut-link has therefore two premises, no conclusion, and the premises cannot be reused for another link.*

Moreover the graph induced by the cut-net should be connected and acyclic.
The conclusion *of the cut-net is $\Xi \vdash \Sigma$, where Ξ, Σ consist of all formulas that are*

not used as premises of cut-links, Ξ listing the main doors and Σ listing the auxiliary doors. It is easy to see that Ξ consists of at most one formula. However we cannot claim that the conclusions form a sequent, since both Ξ and Σ might be empty... and this is indeed the most important case ! When we want to allow $\vdash$ we speak of an extended *sequent.*

The calculus with cuts is defined inductively as follows : a paraproof of a (n extended) sequent consists in a cut-net with this sequent as conclusion, and whose boxes are justified as follows :

1. A box with conclusions $\vdash \Sigma$ *can* be justified by the magical expression *hypothesis.*
2. A box with conclusions $\mathbf{0}^\flat \vdash \Sigma$ *must* be justified by the expression *axiom.*
3. A box with conclusions $\vdash \mathfrak{X}P_1, \ldots, P_n, \Sigma_1, \ldots, \Sigma_k$ *can* be justified by means of a non-empty maximal clique of $\mathfrak{X}$, e.g. $X = \{p_{i_1}, \ldots, p_{i_k}\}$, together with cut-nets whose respective conclusions are $P_{i_1} \vdash \Sigma_1, \ldots, P_{i_k} \vdash \Sigma_k$.
4. A box with conclusions $\mathfrak{X}P_1, \ldots, P_n \vdash \Sigma$ *must* be justified by the data, for any non-empty maximal clique X of $\mathfrak{X}$, e.g. $X = \{p_{i_1}, \ldots, p_{i_k}\}$, of a cut-net whose conclusions are $\vdash \Upsilon_X, \Sigma$, with $\Upsilon_X = P_{i_1}, \ldots, P_{i_k}$.

Observe the difference between « must » (minor sequents) and « can » (major sequents). Remember the distinction between cut-nets (with unjustified boxes), and paraproofs (where the boxes are in turn justified by cut-nets).
The cut-free calculus is the particuliar case where all cut-nets which hereditarily occur are indeed boxes.

1.5 Cut-elimination : eager version

In this process, all cuts will be eliminated. It is done as follows : we select a cut-link anywhere (including in the hereditary justifications of the boxes) and then we proceed according to the case. We now assume that our cut is between two occurrences of ξ, one being an auxiliary door of a box $\mathcal{B}$ ending with the conclusions $\Xi \vdash \Sigma, \xi$ and the other the main door of a box $\mathcal{B}'$ whose conclusions are $\xi \vdash \Sigma'$. The idea is to produce a cut-net whose conclusions are $\Xi \vdash \Sigma, \Sigma'$ and to replace the configuration « Two boxes and the cut-link » with it. In all the minor cases, the cut-net is indeed a box, but in the main case it is a non-trivial cut-net, i.e. some destruction of boxes has been performed.

hypothesis If $\Xi = \emptyset$ and $\mathcal{B}$ is justified as a hypothesis, then we just replace the two boxes with a single one (conclusions $\vdash \Sigma, \Sigma'$), justified as a hypothesis.

axiom If $\Xi = \mathbf{0}^\flat$, then $\mathcal{B}$ is justified as an axiom ; then we just replace the two boxes with a single one (conclusions $\mathbf{0}^\flat \vdash \Sigma, \Sigma'$), justified as an axiom ; observe that, in case our proof was a real one, correctness is respected.

negative commutation If $\Xi = \mathfrak{X}P_1, \ldots, P_n$, then $\mathcal{B}$ is justified by cut-nets $\mathbf{n}_X$, where X varies through all maximal non-empty cliques of $\mathfrak{X}$. Each of these boxes $\mathbf{n}_X$ has a conclusion $\vdash \Sigma, \Upsilon_X, \xi$ and we can enlarge $\mathbf{n}_X$ by means of a cut with the box $\mathcal{B}'$: this yields a cut-net $\mathbf{n}'_X$ whose conclusions are now $\vdash \Sigma, \Upsilon_X, \Sigma'$. Then from the $\mathbf{n}'_X$ we form a box $\mathcal{B}''$ whose conclusions are $\Xi \vdash \Sigma, \Sigma'$, which replaces the previous configuration.

positive commutation If $\Xi = \emptyset$ and $\mathcal{B}$ is justified by cut-nets $\mathbf{n}_1, \ldots, \mathbf{n}_k$, but not with the rule introducing ξ to the right, then one of the $\mathbf{n}_i$ has ξ among its conclusions, say $i = 1$. We cut $\mathbf{n}_1$ with $\mathcal{B}'$, so as to get a cut-net $\mathbf{n}'_1$; then we can form a box with $\mathbf{n}'_1, \mathbf{n}_2, \ldots, \mathbf{n}'_k$, say $\mathcal{B}''$, which replaces the previous configuration.

main case If $\Xi = \emptyset$ and $\mathcal{B}$ is justified by cut-nets $\mathbf{n}_1, \ldots, \mathbf{n}_k$, corresponding to a rule introducing ξ to the right and assume that the respective conclusions of $\mathbf{n}_1, \ldots, \mathbf{n}_k$ are $P_{i_1} \vdash \Sigma_1, \ldots, P_{i_k} \vdash \Sigma_k$, and let $X = \{i_1, \ldots, \imath_k\}$. Now $\mathcal{B}'$ is justified by a bunch of cut-nets, including one for X, say $\mathbf{n}'$, whose conclusions are $\vdash \Sigma', \Upsilon_X$. Υ_X is equal to $P_{i_1}, \ldots, P_{i_k}$, hence we can link the cut-nets $\mathbf{n}_1, \ldots, \mathbf{n}_k$ with $\mathbf{n}'$, yielding a cut-net $\mathbf{n}$" : this cut-net replaces our configuration.

Theorem 1

The procedure just described induces a deterministic full cut-elimination (strong normalization).

PROOF. — It would be out of proportion to prove such an easy result in full details. Let us just give some hints as to the proof :

1. The first thing to check is that all operations preserve the property of being a cut-net : immediate.
2. All possibilities of cut have been considered, so if the algorithm converges, all cuts will be eliminated.
3. The small normalization theorem of [2] can be mimicked : associate a *size* to each box and cut-net, namely :
   (a) $s(\mathcal{B}) = 2$ when $\mathcal{B}$ is axiom or hypothesis.
   (b) $s(\mathcal{B}) = 1 + \sum s(\mathbf{n}_X)$ when $\mathcal{B}$ is obtained by a « left rule ».
   (c) $s(\mathcal{B}) = 1 + (s(\mathbf{n}_{i_1}) \ldots s(\mathbf{n}_{i_k}))$, when $\mathcal{B}$ is obtained by a « right rule ».
   (d) $s(\mathcal{B}) = \prod s(\mathcal{B}_i)$, when $\mathbf{n}$ is obtained from boxes $\mathcal{B}_i$ by means of cut-links.

   It is easy to see that each step makes the size stricly decrease.
4. The last point is Church-Rosser, i.e. the determinism of the computation. This must be obvious to people familiar with general proof-nets [2] or with interaction nets [6]. □

2 Game-theoretic interpretation of HC

2.1 Cut-elimination : lazy case

Since the theorem is based on the decreasing of the size, we can investigate a *lazy* version, namely not to use all our transformations. It will converge, i.e. stop, but not on a cut-free paraproof, especially if we restrict our transformations to the main case, and do not even allow to work inside boxes ; in that case, we can drastically simplify the size as :

1. $s(\mathcal{B}) = 0$ when $\mathcal{B}$ is axiom or hypothesis.
2. $s(\mathcal{B}) = 1 + \sup s(\mathbf{n}_X)$ when $\mathcal{B}$ is obtained by a « left rule ».

3. $s(\mathcal{B}) = 1 + (s(\mathbf{n}_{i_1}) + \ldots + s(\mathbf{n}_{i_k}))$, when $\mathcal{B}$ is obtained by a « right rule ».

4. $s(\mathcal{B}) = \sum s(\mathcal{B}_i)$, when $\mathbf{n}$ is obtained from boxes $\mathcal{B}_i$ by means of cut-links.

and we rediscover the familiar linearity of lazy evaluation in the absence of exponentials.
But the question is to determine whether this is still interesting. For this we shall only consider cut-nets with an empty conclusion $\vdash$. The main point is that in such a cut-net, at most one cut-link can be simplified : indeed there is exactly one (external) box which has no main door, say $\vdash \Sigma$, and this box is a hypothesis or it is justified by a right introduction of $\xi = \mathfrak{X}P_1, \ldots, P_n$ in which case this formula ξ is linked with a main door of a box $\mathfrak{X}P_1, \ldots, P_n \vdash \Sigma'$, which is also justified by a (here : left) introduction of the same ξ. Then the main case applies, and we are left with a similar configuration... this procedure is purely sequential, i.e. only one transformation can be performed at each step. When the procedure stops, this is always because the major sequent is badly justified (hypothesis) : otherwise, the formula ξ introduced to the right is distinct from $\mathbf{0}^\flat$, hence we are in the main case.
Now we decide to consider the cut-elimination steps as the moves in a particular play. Our first concern is to individuate the players's moves. For this, let us start with the simplest case of a cut between $P \vdash$ and $\vdash P$. After the first step we shall get —say— three boxes ($\vdash P_1, P_5$; $P_1 \vdash$ and $P_5 \vdash$) with two cut-links, P_1, P_5 being immediate subformulas of P. After the second step one of the P_i, say P_5 will be split into —say— three immediate subformulas, say P_{52}, P_{55}, P_{57}, and the cut-net will now consist of five boxes ($P_{52} \vdash$; $P_{55} \vdash$; $P_{57} \vdash P_1$; $P_1 \vdash$; $\vdash P_{52}, P_{55}, P_{57}$) with four cut-links... Now associate a *parity* to all the hereditary subformulas of P, so that P is even, P_5 is odd, P_{57} is even etc., i.e. the parity of the depth of the subformula. Then one checks that :

1. Each sequent occurring at any stage in the process (i.e. any external box) has the property that the right-hand side has s definite parity, which is opposite to the parity of the left-hand side (if this makes sens). From this we can define a parity for any sequent, e.g. $P_{52} \vdash$ is odd as well as $P_{57} \vdash P_1$ and $\vdash P_1, P_5$, whereas $\vdash P$ and $\vdash P_{52}, P_{55}, P_{57}$ are even.

2. Each of the external cut-nets has a unique major sequent, and we observe that the parity alternates. The first position is even, the second is odd, the third is even etc. We can therefore see the process as the result of a *dispute* between the players Even and Odd.

3. If one the players stops, i.e. if cut-elimination cannot proceed further, it must be because one of the players didn't play, and this corresponds to a hypothesis. Clearly this player loses. But what about the axiom for $\mathbf{0}^\flat$? Indeed a configuration $\mathbf{0}^\flat \vdash \Sigma_i$ can be created by an appropriate move from a configuration $\vdash \Sigma, Q$, and Σ_i is now captive, in the sense that, since $\mathbf{0}^\flat$ cannot be activated, (remember : there is no right rule for $\mathbf{0}^\flat$!), it will never be possible to proceed further. Thus the fact of giving part of the context Σ to $\mathbf{0}^\flat$ is approximately the same as erasing Σ_i, and of course, erasing Σ_i may help a lot. This is why this move is considered as *unfair*, more precisely to be a *dog's move*. There is however one exception : if Σ_i only consists of occurrences of $\mathbf{0}^\flat$, we can hardly say that some damage has been caused, since this part of the context was already frozen for ever. We can summarize this by declaring that there two ways of losing :

   give up Just stop playing (hypothesis).

   dog's move When playing, create a sequent $\mathbf{0}^\flat \vdash \Sigma, P$, with $P \neq \mathbf{0}^\flat$.

2.2 Soundness

This is not yet the real thing, but just the remark that :

- A paraproof $\mathbf{n}$ of $\vdash P$ and a paraproof $\mathbf{m}$ of $P \vdash$ generate a finite dispute, noted $\mathbf{nm}$, or $\mathbf{mn}$ (we prefer not to distinguish in this notation between Even and Odd).
- If —say— $\mathbf{n}$ uses no paralogism, then Even wins the dispute $\mathbf{nm}$.

As expected, soundness is basically a reformulation of cut-elimination.

2.3 Open your box, sir !

But we shall have some difficulty to prove a corresponding first approximation to completeness. We ask for something very weak indeed : given a paraproof $\mathbf{n}$ of a sequent —say $\vdash P$—, is it possible to recover it from all disputes $\mathbf{nm}$, when $\mathbf{m}$ varies through all paraproofs of $P \vdash$? The difficulty is with unopened boxes : if a player can force his opponent to open his boxes in the right way, then he is done.
Assume that Even plays following the paraproof $\mathbf{n}$, and that Odd wants to reconstruct it.

- Even replaces $\vdash P$ with $P_1 \vdash$, $P_5 \vdash$, and Odd, obeying *perinde ac cadaver* replaces his own $P \vdash$ with $\vdash P_1, P_5$. Odd now must decide to check the paraproof above $P_1 \vdash$ or $P_5 \vdash$.
- He decides to see $P_5 \vdash$; he knows that anyway the last rule of Even is the rule of a specific connective $\mathfrak{X}$, whose premises are of the form $\vdash \Upsilon_X$. He decides to check above a specific $\vdash \Upsilon_X$, and he chooses —say— $X = \{2,5,7\}$. Then Even obeys in turn like a Jesuit and the new major sequent $\vdash P_{52}, P_{55}, P_{57}$ is created...
- Then Even must play or give up. If he plays, then Odd goes on as before.
- Eventually, a full branch of the tree will be explored.

Indeed, the exploration just described corresponds to a specific $\mathbf{nm}$: the paraproof $\mathbf{m}$ starts with a left-introduction rule, corresponding to a specific synthetic connective, it must have a certain number of premises, including $\vdash \Upsilon_{\{1,5\}}$, that we have actually used... but what about the premise —say— $\vdash \Upsilon_{\{4,6,9\}}$, if such a premise is needed ? This is simple, Odd gives up (admits it as a hypothesis). In other terms, Odd can explore paraproofs only because there are enough paralogisms, here « give up ».
Then the full structure can be reconstructed by allowing Odd to make other trials, i.e. the paraproof $\mathbf{n}$ can been reconstructed from the disputes $\mathbf{nm}$. But there is still a pending question : if Even wins all disputes $\mathbf{nm}$, does this mean that $\mathbf{n}$ is a real (i.e. correct) proof ? In fact a strange phenomenon might perhaps happen when Odd explores a branch : at some moment he couldn't choose a move (i.e. a right introduction) corresponding to the formula Q he would like to explore, to the effect that Even would win, but perhaps only for bad reasons. In fact there could only be two possibilities :

1. Q begins with $\mathfrak{X}$, but there is no rule at all introducing Q the right : we are in the case of the constant $\mathbf{0}$, the synthetic 0-ary connective. Odd would like to question $\mathbf{0} \vdash \Sigma$ in $\mathbf{n}$, he cannot, but anyway this sequent is provable by a left introduction (no premise at all).
2. Q is $\mathbf{0}^\flat$ and remember that there is no rule introducing $\mathbf{0}^\flat$ to the right. But in this case, Odd wants to check something like $\mathbf{0}^\flat \vdash \Sigma$ in $\mathbf{n}$. But the presence of such a sequent in

$\mathbf{n}$ can only be attributed to dog's play, unless Σ is made of occurrences of $\mathbf{0}^\flat$. In other terms, either Even has already lost and Odd is wasting his time, or $\mathbf{0}^\flat \vdash \Sigma$ is a correct axiom.

Since our potential objection didn't work, $\mathbf{n}$ is a real proof as an easy consequence of the additional remark that $\mathbf{n}$ is finite.
Let us reflect a little more on dog's play : assume that Even wants to prove $\bot \otimes \bot$. Then he starts splitting the tensor, and we get the configuration $P \vdash$, $Q \vdash$ (for Even) against $\vdash P, Q$ (for Odd), with $P = Q = \mathbf{1}$. Now if the poor Odd cannot play the dog, he must give up, and Even has won too easily. So Odd will be allowed to play —say— $\mathbf{0}^\flat \vdash P$ and Even will receive $\vdash \mathbf{0}^\flat$, on which he must give up, hence he loses. Now it would be excessive to give the victory to Odd : since he « killed P » he loses too.
Dog's play is not quite using a wrong rule, but using a rule in such a way that one premise cannot be decently proven : typically, prove $\vdash P^\perp \otimes \mathbf{1}, \mathbf{1}$ from $P \vdash$ and $\mathbf{0}^\flat \vdash \mathbf{1}$ is using a correct rule... but introducing an « impossible » premise.

2.4 Projection

We now come to the syntactical aspect of the most important artefact, namely *projection*.
Assume for instance that our starting configuration is $P \vdash$, $Q \vdash$ (for Even) against $\vdash P, Q$ (for Odd). Now the dispute proceeds ; but would it be possible to produce a relativized dispute, i.e. a dispute between $P \vdash$, and $\vdash P$?. The answer is YES.
Let $\mathfrak{m}$ be one of the configurations of the dispute ; the formulas occurring in $\mathfrak{m}$ are either subformulas of P or subformulas of Q. We decide to eliminate all subformulas $\mathbf{q}$ of Q as follows : if we find two boxes $\Xi \vdash \Sigma, \mathbf{q}$ and $\mathbf{q} \vdash \Sigma'$, we replace them with $\Xi \vdash \Sigma, \Sigma'$. We do this *ad nauseam* until we get a configuration made only of subformulas of P : these configurations are the *projections* on P of the original ones.
Now consider any move, making us pass from a configuration to another. They fall into 4 classes, for which we indicate what happens to the respective projections of the initial and final configurations.

P-Even (active) Even plays, with the focus in a subformula of P. Then this can mimicked by the same move.

Q-Even (passive) Even plays, with the focus in a subformula of Q. This leaves no trace on the projections.

Q-Odd (passive) Odd plays, with the focus in a subformula of Q. This move is not visible on the projection.

P-Odd (active) Odd plays, with the focus on a subformula of P. Then it is possible to find a move of Odd, with the same focus etc., which acts in the right way on the projections.

In other terms a dispute can be projected as a dispute.
Projection is associative : if we had started with $P \vdash$ and $Q \vdash$ and $R \vdash$ (for Even) against $\vdash P, Q, R$ (for Odd), then we could have projected on P in three ways, either directly, or by first projecting on P, Q or on P, R : the three methods yield the same projection.
Projection is indeed the only convincing way to explain eager cut-elimination :

Theorem 2

Let $\mathbf{l}, \mathbf{n}, \mathbf{m}$ *be respective cut-free paraproofs of* $\vdash P, Q$ *and* $P \vdash$ *and* $Q \vdash$ *; then we can consider on one side the dispute* $\mathbf{lnm}$ *obtained by means of two cut-links, and project it on* P*, so as to get a dispute* $\mathbf{lnm}\lceil P$*. We can also make a single cut-link on* Q *and obtain through eager cut-elimination, a cut-free paraproof* $\mathbf{l}(\mathbf{m})$ *of* $\vdash P$*, and we can therefore produce a dispute* $\mathbf{l}(\mathbf{m})\mathbf{n}$*. Then*

$$\mathbf{lnm}\lceil P = \mathbf{l}(\mathbf{m})\mathbf{n}$$

Proof. — Straightforward. □

2.5 The game-theoretic translation

We are almost done with this side of the tunnel. The ultimate thing is to remove syntax : for this we decide to specialize our construction w.r.t. a formula P of **HC**, and our results —completeness, soundness— will refer to paraproofs of $\vdash P$ or $P \vdash$.

We now consider the full tree of finite sequences of integers : I represent each subformula of P by an element of the tree, typically $P =< 0 >$. If a subformula $Q = \mathfrak{X}R_1 \ldots R_n$ is represented by s, then we choose distinct indices $i_1, \ldots, i_n$, and represent R_1 by $s * i_1$, R_n by $s * i_n$. The parity will correspond to the length of the sequence, i.e. $< i_0, \ldots, i_n >$ has the parity of n.

In fact we are not completely free in the choice of the indices : we must first split $\mathbb{N}$ into two infinite subsets (even numbers and odd numbers if you want, but call them « simple » and « multiple »). Now, i_k will be chosen simple when $R_k \neq \mathbf{0}^\flat$, multiple otherwise.

Now our analysis led us to the point when the disputes $\mathbf{nm}$ can be described as particular plays in a game involving this tree. Our problem is now to find an alternative description of the same thing.

3 Disputes

3.1 Fingers, Hands and Blokes

Definition 7

Fingers *are the elements of a denumerable set, say* $\mathbb{N}$ *; we distinguish between* simple *and* multiple *fingers, i.e. by writing* $\mathbb{N}$ *as the union of two infinite disjoint subsets, e.g. even and odd numbers.*

$\mathbb{H}$ *is the tree of all finite sequences* $<x_0, \ldots, x_{n-1}>$*, where* $x_i \in \mathbb{N}$ *are fingers. The elements of* $\mathbb{H}$ *of length* $\neq 0$ *are called* hands *; according to the parity of* $n-1$ *a hand is* even *or* odd*. If we use* « $*$ » *for concatenation, then* $\sigma * \tau \leqslant \sigma$ *defines an order, the* subhand *relation : if* $\tau \neq <>$ *one speaks of a* strict *subhand, if* $\tau = <i>$*, of an* immediate *subhand ; in that case parity alternates, e.g. an immediate subhand* $\sigma * i$ *of an even hand* σ *is odd.*

The hands $s * i$ *with* i *simple are* simple*, whereas hands* $s * i$ *with* i *multiple are* multiple.

Definition 8

A bloke $\Xi \vdash \Sigma$ *consists in :*

1. *Two finite sets of hands* Ξ, Σ *; these hands must be pairwise incomparable and* $\Xi \cup \Sigma$ *must be non-empty*

2. *The hands in* Σ *have the same parity ;*
3. Ξ *is either empty or consists of a single hand* ξ *of parity opposite to* Σ *(if non empty) ; in the first case the bloke is* major, *otherwise it is* minor *; the hand* ξ *of the minor bloke* $\mathfrak{b} = \xi \vdash \Sigma$ *is the* tutor *of* $\mathfrak{b}$.

The common parity of Σ *and/or the parity opposite to* Ξ *is the* parity *of the bloke ; there are therefore even blokes and odd blokes.*
A bloke $\xi \vdash \Sigma$ *with* ξ *multiple is said to be* multiple *; other blokes are* simple. *A bloke is* schizoid *when it is multiple, but its right-hand side contains at least one simple hand.*

We shall use the familiar conventions of sequent calculus, e.g. Σ, Σ' for (disjoint) union, σ instead of $\{\sigma\}$, so that we can write $\vdash \Sigma, \sigma$ or $\xi \vdash \Sigma, \sigma$ etc.

3.2 Mauls

Definition 9

The full maul $\mathbb{M}$ *consists in all finite sequences of hands* $<\sigma_0, \ldots, \sigma_{n-1}>$. *This tree is ordered as usual by :* $<\sigma_0, \ldots, \sigma_{n+m-1}> \leqslant <\sigma_0, \ldots, \sigma_{n-1}>$.

Definition 10

An even maul $\mathfrak{m}$ *of is a finite subtree of* $\mathbb{M}$ *such that :*

1. $\mathfrak{m}$ *is non-trivial, i.e. non restricted to its root* $<>$
2. *If a hand* σ *occurs in* $\mathfrak{m}$, *i.e. belongs to some* $<\sigma_0, \ldots, \sigma_{n-1}, \sigma> \in \mathfrak{m}$, *then its parity is equal to parity of* n.
3. *The hands occurring in* $\mathfrak{m}$ *are pairwise incomparable, in particular there is at most one sequence* $<\sigma_0, \ldots, \sigma_{n-1}, \sigma> \in \mathfrak{m}$ *ending with* σ.

An odd maul is defined in the same way, except that the parity of σ *such that* $<\sigma_0, \ldots, \sigma_{n-1}, \sigma> \in \mathfrak{m}$ *is opposite to the parity of* n.

In particular, we can associate blokes to a maul $\mathfrak{m}$, namely :

1. The bloke $\vdash \Sigma$, with $\Sigma = \{\sigma; <\sigma> \in \mathfrak{m}\}$; this bloke is the unique *major bloke* of $\mathfrak{m}$. The *parity* of $\mathfrak{m}$ is equal to the parity of its major bloke.
2. For each ξ occurring in $\mathfrak{m}$ through the sequence $\ell = <\sigma_0, \ldots, \sigma_{n-1}, \xi> \in \mathfrak{m}$, the bloke $\xi \vdash \Sigma$, with $\Sigma = \{\sigma; \ell * \sigma \in \mathfrak{m}\}$. These are the *minor blokes* of $\mathfrak{m}$.

Conversely, observe that a maul $\mathfrak{m}$ can be recovered from its blokes : $<\sigma_0, \ldots, \sigma_n> \in \mathfrak{m}$ iff σ_0 belongs to the major bloke of $\mathfrak{m}$ and for each $i < n$ there is a minor bloke $\sigma_i \vdash \Sigma$ in $\mathfrak{m}$ with $\sigma_{i+1} \in \Sigma$.

Example 1

Given a bloke $\Xi \vdash \Sigma$ *we can form a maul by adding the atomic blokes* $\sigma \vdash$ *(for* $\sigma \in \Sigma$*) and* $\vdash \xi$ *(for* $\xi \in \Xi$*). This maul is the* standard maul *associated with* $\Xi \vdash \Sigma$, *and we shall abusively note it* $\Xi \vdash \Sigma$.

Remark. — An alternative presentation of a maul is a forest of hands (a *forest* is a partial order such that for each x the set of points greater than x is linearily ordered). In that case the major bloke is the set of maximal elements of the forest, whereas the minor bloke $\xi \vdash \Sigma$ gives the set Σ of immediate predecessors of ξ.

It is sometimes convenient to introduce the *unproper maul*, namely the trivial tree reduced to its root.

3.3 Actions and Disputes

Definition 11

Let $\mathfrak{m}$ *be a maul, with major bloke* $\vdash \Sigma$ *; an* action κ *in* $\mathfrak{m}$ *consists in the following data :*

1. *A hand* $\xi \in \Sigma$ *, the* focus *of the action ; we can therefore write* $\Sigma = \Sigma', \xi$
2. *A non-empty finite subset* I *of* $\mathbb{N}$ *; let* $\Xi = \{\xi * i ; i \in I\}$
3. *For each* $i \in I$ *a bloke* $\xi * i \vdash \Sigma_i$ *; we require* Σ' *to be the disjoint union of the* Σ_i*.*

Definition 12

The result $\mathfrak{m}' = \kappa(\mathfrak{m})$ *of the action* κ *is obtained as follows :* $\mathfrak{m}$ *has a unique bloke of the form* $\xi \vdash \Upsilon$ *; then* $\kappa(\mathfrak{m})$ *has exactly the same blokes as* $\mathfrak{m}'$*, except that* $\vdash \Sigma$ *is replaced with the* $\xi * i \vdash \Sigma_i$ *and* $\xi \vdash \Upsilon$ *is replaced with* $\vdash \Upsilon, \Xi$.

A direct description of $\mathfrak{m}'$ is the set of :

1. For any sequence $<\xi, \sigma_1, \ldots, \sigma_n> \in \mathfrak{m}$, the sequence $<\sigma_1, \ldots, \sigma_n>$
2. For any sequence $<\sigma_0, \ldots, \sigma_n> \in \mathfrak{m}$, with $\sigma_0 \in \Sigma_i$, the sequence $<\xi_i, \sigma_0, \ldots, \sigma_n>$.

An action changes the parity of the maul. In images, if $\mathfrak{m}$ is even, then Even acts, and produces $\mathfrak{m}'$ which is odd ; then Odd may act etc.

Definition 13

A dispute *is a finite sequence* $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p]$*, such that* $\mathfrak{m}_{i+1} = \kappa_i(\mathfrak{m}_i)$ *for appropriate actions* κ_i*.*
$\mathfrak{m}_0$ *is the* initial *maul of the action,* $\mathfrak{m}_p$ *its* final *maul ;* n *is the* length *(duration ?) of the dispute.*

The sequence of actions can be recovered from the sequence of mauls : this is why we never indicate the actions of a dispute.

3.4 Output of a dispute

Definition 14

A dispute $\mathfrak{D}$ *is* lost *by Odd when one of the following holds :*

schizoid bloke *One of the odd blokes is of the form* $\xi \vdash \Sigma, \eta$*, with* ξ *multiple and* η *simple.*

give up *The last maul is odd.*

We define similarly the notion of dispute lost by Even. A dispute has necessarily at least one loser, and by complement, at most one winner*.*

In general the starting maul is simple and contains no multiple hand. Then a schizoid odd bloke $\xi \vdash \Sigma, \eta$, with ξ multiple and η simple must have been created by an action of Odd, what we call a *dog's move.*
In order to win, Even must always be fair play, and should not give up, i.e. must be the last to play. Of course unfair play will often be the only way to avoid giving up, at the price of a draw. This is why we speak of « dog's move », see [3].

3.5 Projections

Definition 15

Let $X, Y \subset \mathbb{H}$ be two disjoint sets of hands closed under the subhand relation. X, Y is said to realize an even splitting *of a maul $\mathfrak{m}$ when the following holds :*

1. *Every hand occurring in $\mathfrak{m}$ belongs to $X \cup Y$*
2. *If an even bloke of $\mathfrak{m}$ contains a hand of X, then all hands of the bloke are included in X. Odd splittings are defined in the same way.*

A splitting is non-trivial *when both X and Y meet $\mathfrak{m}$.*

Example 2

An even bloke of $\mathfrak{m}$ determines an even splitting : let X be the set of subhands of the bloke and Y be the set of subhands of the other even blokes.

Definition 16

To any sequence in $\ell \in \mathfrak{m}$ we associate the projected *sequence $\ell \upharpoonright X$ in which only the elements of X remain. Then the* projection *of the maul $\mathfrak{m}$ on X is defined as $\mathfrak{m} \upharpoonright X = \{\ell \upharpoonright X; \ell \in \mathfrak{m}\}$. Particular cases : projection on a bloke $\mathfrak{b}$ of $\mathfrak{m}$, noted $\mathfrak{m} \upharpoonright \mathfrak{b}$, and in case of an atomic bloke $\xi \vdash$ or $\vdash \xi$ of $\mathfrak{m}$, projection on the hand ξ, noted $\mathfrak{m}$, $\mathfrak{m} \upharpoonright \xi$.*

Another description of projection is by iterated removals, i.e. « use of cut » : removing one hand σ from a maul consists in replacing the two blokes containing σ, i.e. $\Xi \vdash \Sigma, \sigma$ and $\sigma \vdash \Upsilon$ by $\Xi \vdash \Sigma, \Upsilon$ (which need not be a bloke since it might be empty or have no definite parity). Removing several hands is the iteration of the previous algorithm ; the result does not depend on the order of the removals. Projecting $\mathfrak{m}$ on X consists in removing all hands which do not belong to X.

Proposition 3

Assume that X, Y realize a non-trivial splitting (even or odd) of $\mathfrak{m}$; then $\mathfrak{m} \upharpoonright X$ is a maul.

PROOF. — The splitting condition ensures that, whenever σ disappears, then a full bloke containing σ disappears too. The result therefore follows from the fact that the removal of a full bloke $\Xi \vdash \Sigma$ preserves the fact of being a maul (maybe unproper) ; in fact all blokes containing one of the hands of the —say— even bloke $\Xi \vdash \Sigma$ are replaced with an odd bloke. If the splitting is non-trivial, we get a proper maul. □

Proposition 4

Assume that X, Y realize a non-trivial splitting (even or odd) of $\mathfrak{m}$, and let $\mathfrak{m}' = \mathfrak{m} \upharpoonright X$. Then one can define for each hand ξ occurring in $\mathfrak{m}$ a finite set $X(\xi)$ of hands of X with the following property : if we extend the map to sets of hands by $X(\Sigma) = \bigcup_{\sigma \in \Sigma} X(\sigma)$, then

1. *If $\vdash \Sigma$ is the major bloke of $\mathfrak{m}$ then $\vdash X(\Sigma)$ is the major bloke of $\mathfrak{m}'$*
2. *If $\xi \in X$ and $\xi \vdash \Sigma$ is a minor bloke in $\mathfrak{m}$, then $\xi \vdash X(\Sigma)$ is a minor bloke in $\mathfrak{m}'$*
3. *If $\xi \notin X$ and $\xi \vdash \Sigma$ is a minor bloke in $\mathfrak{m}$, then $X(\xi) = X(\Sigma)$*

PROOF. — If $\ell * \xi \in \mathfrak{m}$, then we can consider all its extensions $\ell' * \sigma \in \mathfrak{m}$ such that $\sigma \in X$ and which are maximal, i.e. such that no subextension ends in X ; $X(\xi)$ is defined as the set of the hands σ obtained in this way. The properties are obvious. □

Proposition 5

If X, Y realize an even splitting of $\mathfrak{m}$ and $\mathfrak{m}'$ is the result of an action from $\mathfrak{m}$, then X, Y realize an even splitting of $\mathfrak{m}'$.

PROOF. — Each new bloke in $\mathfrak{m}'$ is made of « subblokes » of a bloke of $\mathfrak{m}$ with the same parity. This is enough for the result. □

We investigate the behavior of an action $\kappa = (\xi, \Sigma_i)$ under non-trivial projection :

X-Even If the focus ξ of κ is in X and $\mathfrak{m}$ is even, then the major bloke of $\mathfrak{m}$ is included in X and $\mathfrak{m}' \upharpoonright X = \kappa(\mathfrak{m} \upharpoonright X)$

X-Odd If the focus ξ of κ is in X and $\mathfrak{m}$ is odd, then projection replaces the major bloke $\vdash \Sigma', \xi$ by $\vdash X(\Sigma'), \xi$. If κ' is the action with the same focus, the same I, but with $X(\Sigma_i)$ instead of Σ_i, then clearly $\mathfrak{m}' \upharpoonright X = \kappa'(\mathfrak{m} \upharpoonright X)$.

Y-Even If the focus of κ is not in X and $\mathfrak{m}$ is even, then $\mathfrak{m}' \upharpoonright X = \mathfrak{m} \upharpoonright X$ and the action induces nothing.

Y-Odd If the focus of κ is not in X and $\mathfrak{m}$ is odd : as above, nothing.

In other terms an action κ is *projected* into $\kappa \upharpoonright X$, which is either an action or nothing.
In case of consecutive actions, observe that, due to alternation of parities, cases *X-Even* or *Y-Even* can only be followed by cases *X-Odd* or *Y-Odd* ; conversely case *X-Odd* can only be followed by case *X-Even*, and case *Y-Odd* can only be followed by case *Y-Even*.
Let $\mathfrak{D} = [\mathfrak{m}_0, \ldots, \mathfrak{m}_p]$ be a dispute and assume that X, Y realize a non-trivial splitting (—say even—) of the initial maul ; then we can consider $[\mathfrak{m}_0 \upharpoonright X, \ldots, \mathfrak{m}_p \upharpoonright X]$ which is a sequence of mauls. Each $\kappa_i \upharpoonright X$ is either an action or nothing ; and let $i_1, \ldots, i_k$ is the increasing list of those which are actions. Then, between two consecutive mauls of the sequence $[\mathfrak{m}_0 \upharpoonright X, \mathfrak{m}_{i_1+1} \upharpoonright X, \ldots, \mathfrak{m}_{i_k+1} \upharpoonright X]$, only action κ_k matters ; this sequence is therefore a dispute, and we have indeed proved the :

Proposition 6

The sequence $[\mathfrak{m}_0 \upharpoonright X, \mathfrak{m}_{i_1+1} \upharpoonright X, \ldots, \mathfrak{m}_{i_k+1} \upharpoonright X]$, where $i_1, \ldots, i_k$ is the subsequence made of those i such that the focus of κ_i belongs to X, is a dispute, called the projection *of $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p]$. Notation : $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p] \upharpoonright X$. Particular case : projection on a bloke of $\mathfrak{m}_0$, on a hand etc.*

Theorem 3

Let $\mathfrak{D}$ be a dispute, let X, Y realize a non-trivial even splitting of its initial maul $\mathfrak{m}$ and let $\mathfrak{E}$, $\mathfrak{F}$ be the respective projections of the disputes on X and Y then :

1. *If Even wins $\mathfrak{D}$, then it wins both $\mathfrak{E}$ and $\mathfrak{F}$.*
2. *If Even wins both $\mathfrak{E}$ and $\mathfrak{F}$, then it wins $\mathfrak{D}$.*
3. *If Odd wins $\mathfrak{D}$ and Even wins the projection $\mathfrak{E}$, then Odd wins the projection $\mathfrak{F}$.*

PROOF. —

1. Since Even wins, all even blokes are fair ; moreover, in the projected dispute, even blokes disappear or remain the same. Let us now look at the last maul of the projected dispute : if it is even, then its major bloke is in X and it comes from an even maul in the original dispute, i.e. a kind *X-Even*, but since Even wins, it cannot be the final maul of the original dispute.

2. Similar argument.

3. An odd bloke in $\mathfrak{F}$ comes from an odd bloke in the original dispute. Assume this bloke to be unfair : then it is of the form $\xi \vdash \Sigma, \eta$, with ξ multiple and η simple, and it has been obtained by iterated cuts, starting with an odd bloke $\xi \vdash \Sigma'$, with even blokes (but located in X) and other odd blokes. Since Even wins the projection on X, all these blokes are fair, i.e. the result of the cuts is fair. If the last maul of the dispute $\mathfrak{F}$ is odd, then it comes from an odd maul in the original dispute (impossible, since it won by Odd) or from an even maul with major bloke in X (impossible, since Even wins the projection on X). □

4 Designs and Behaviors

4.1 Chronicles and agendas

Our basic idea is to present a strategy $\mathcal{S}$ as the set of all disputes $\mathcal{S}\mathcal{T}$ generated by a counterstrategy $\mathcal{T}$. For this we shall equip disputes with a structure of coherent space, and try to define a strategy as any *clique* of the space ; since a clique and an anticlique intersect on at most one point, this seems to be promising. Now what could be the notion of coherence ? This is simple : if two disputes are distinct, it is because one the two players made a different action, and even coherence should correspond to the case where Odd was the first one to change his mind. This notion has the nice property of defining an unambiguous first action, in case the initial maul is even… But it doesn't work : the notion is not compatible with projection. To understand this essential point, take the example of two disputes $\mathfrak{D}$, $\mathfrak{E}$, which first differ on an action $Y - Odd$; when I project them on X, this part cannot be seen (the view ?, see below), which shows that Even cannot take this as a pretext to play different actions of kind *X-Even* afterwards. This is why the notion of coherence must be localized, and this leads to chronicles and agendas. By the way we are not quite producing a coherent space, since $\mathfrak{D}$, $\mathfrak{E}$ might be neither evenly or oddly coherent.
The notions defined below are presumably related to the *views* of « H^2O-games » introduced by **H**anno Nickau [7] and **H**yland & **O**ng [5].

Definition 17

Let $\mathfrak{D} = [\mathfrak{m}_0, \ldots, \mathfrak{m}_p]$ *be a dispute, and let* $\mathbf{b}$ *be a bloke in* $\mathfrak{m}_{i+1}$ *which is not in* $\mathfrak{m}_i$ *(there is at most one such* i*), then* $\mathbf{b}$ *has a well-defined* father *in* $\mathfrak{m}_i$*, namely a bloke* $\mathbf{a}$ *of the same parity (*$\mathbf{b}$ *is said to be a* son *of* $\mathbf{a}$*) : with the notation of definition 12 for the action* κ_i

1. *If* $\mathbf{b}$ *is* $\vdash \Upsilon, \Xi$*,* $\mathbf{a}$ *is* $\xi \vdash \Upsilon$
2. *If* $\mathbf{b}$ *is* $\xi_i \vdash \Sigma_i$*,* $\mathbf{a}$ *is* $\vdash \Sigma, \xi$

A chronicle *is a sequence* $<\mathbf{a}_0; \ldots; \mathbf{a}_{k-1}>$ *of blokes such that* $\mathbf{a}_i$ *is the father of* $\mathbf{a}_{i+1}$ *for each* $i < n$*, and such that, in case* $k \neq 0$*,* $\mathbf{a}_0$ *is a bloke of* $\mathfrak{m}_0$*, in which case it is called the chronicle of* $\mathbf{a}_{k-1}$*. Each non-zero chronicle has a well-defined parity, and we decide to give both parities to the empty chronicle.*
The even agenda *of* $\mathfrak{D}$ *is the tree of its even chronicles. The odd agenda is defined in the same way.*

Proposition 7

A dispute is completely determined by its agendas (both of them are needed).

Proof. — The chronicles of length 1 give the blokes of $\mathfrak{m}_0$; then if the initial maul is —say odd—, with major bloke $\mathbf{a}$, there is a first action exactly when there is a chronicle of length 2, $<\mathbf{a}; \xi * i \vdash \Sigma_i>$, in which case we know the focus ξ of this rule. Consider all other chronicles of length 2 starting with $\mathbf{a}$; they are of the form $<\mathbf{a}; \xi * i \vdash \Sigma_i>$, which completely determines the first action, etc. □

4.2 Coherence

Definition 18

Two disputes $\mathfrak{D} = [\mathfrak{n}_0, \ldots, \mathfrak{n}_p]$ *and* $\mathfrak{C} = [\mathfrak{m}_0, \ldots, \mathfrak{m}_q]$ *are* evenly coherent *when the following holds :* $\mathfrak{n}_0 = \mathfrak{m}_0$ *and whenever an even bloke* $\vdash \Sigma$ *occurs in both disputes with the same chronicle, then it has the same sons in both disputes. Odd coherence is defined in a similar way.*

Proposition 8

If two disputes $[\mathfrak{n}_0, \ldots, \mathfrak{n}_p]$ *and* $[\mathfrak{m}_0, \ldots, \mathfrak{m}_q]$ *share the same initial action* κ*, i.e.* $\mathfrak{n}_1 = \mathfrak{m}_1$*, then they are evenly coherent iff their continuations* $[\mathfrak{n}_1, \ldots, \mathfrak{n}_p]$ *and* $[\mathfrak{m}_1, \ldots, \mathfrak{m}_q]$ *are evenly coherent.*

Proof. — Immediate. □

Proposition 9

If two disputes $\mathfrak{D}$ *and* $\mathfrak{C}$ *start with an even maul and are evenly coherent, then either they are both of length 0, or they share the same first action.*

Proof. — If $\mathfrak{D}$ has a first action κ, then its even agenda contains all chronicles $<\vdash \Sigma, \xi; \xi * i \vdash \Sigma_i>$ and $<\vdash \Sigma, \xi; \vdash \Sigma>$ corresponding to κ, and since the even agenda of $\mathfrak{C}$ contains $<\vdash \Sigma, \xi>$, all these chronicles must belong to the even agenda of $\mathfrak{C}$. This forces the existence of a first action in $\mathfrak{C}$, with the same focus ξ, etc. □

Corollary 4

Two disputes are both evenly and oddly coherent iff they are equal.

Proposition 10

If two disputes $\mathfrak{D}$ and $\mathfrak{E} = [\mathfrak{m}_0, \ldots, \mathfrak{m}_p]$ are evenly coherent and $q < p$ is such that $\mathfrak{m}_q$ is odd, then $\mathfrak{D}$ and $\mathfrak{F} = [\mathfrak{m}_0, \ldots, \mathfrak{m}_q]$ are evenly coherent.

Proof. — The restriction to p means that some chronicles of $\mathfrak{E}$ are shortened. So let us consider a bloke **a** whose chronicles belongs to the even agendas of $\mathfrak{D}$ and $\mathfrak{F}$; then it has the same sons in the even agenda of $\mathfrak{D}$ and $\mathfrak{E}$. If **a** has a son in the even agenda of $\mathfrak{E}$, observe that the action producing this son must be done without delay, i.e. it is a κ_q with $q \leqslant p$, and since $p = q$ is impossible for questions of parity, the son is still in the agenda of $\mathfrak{F}$. □

Corollary 5

If $\mathfrak{m}$ is even, then the trivial dispute « give up » $[\mathfrak{m}]$ is evenly coherent only with itself ; if $\mathfrak{m}$ is odd, it is evenly coherent with anybody.

The next result expounds the behavior of coherence under projection.

Theorem 6

Let $\mathfrak{D} = [\mathfrak{n}_0, \ldots, \mathfrak{n}_p]$ and $\mathfrak{E} = [\mathfrak{m}_0, \ldots, \mathfrak{m}_q]$ be two disputes with the same initial maul $\mathfrak{n}_0 = \mathfrak{m}_0 = \mathfrak{m}$. Assume that X, Y realizes a non-trivial even splitting of $\mathfrak{m}$, and let $\mathfrak{D}'$, $\mathfrak{D}$", $\mathfrak{E}'$ and $\mathfrak{E}''$ be the projections $\mathfrak{D}\lceil X$, $\mathfrak{D}\lceil Y$, $\mathfrak{E}\lceil X$, $\mathfrak{E}\lceil Y$. Then :

1. *If $\mathfrak{D}$ and $\mathfrak{E}$ are evenly coherent, so are $\mathfrak{D}'$ and $\mathfrak{E}'$.*
2. *Conversely if $\mathfrak{D}'$ and $\mathfrak{E}'$ are evenly coherent, as well as $\mathfrak{D}''$ and $\mathfrak{E}''$, so are $\mathfrak{D}$ and $\mathfrak{E}$.*
3. *If $\mathfrak{D}$ and $\mathfrak{E}$ are oddly coherent, if $\mathfrak{D}'$ and $\mathfrak{E}'$ are evenly coherent, then $\mathfrak{D}''$ and $\mathfrak{E}''$ are oddly coherent.*
4. *Moreover, in case 3, if $\mathfrak{D}$ and $\mathfrak{E}$ are distinct so are their projections $\mathfrak{D}''$ and $\mathfrak{E}''$.*

Proof. — All proofs are by induction on $\inf(m, n)$:

1. Due to splitting, the even agendas of the projections are just the subtrees containing only the chronicles starting with a bloke in X.
2. Similar argument.
3. We first consider the case where $\mathfrak{m}\lceil Y$ has only one odd bloke **a** : in particular the odd agendas of the projections on Y will have only one chronicle of length one. If the dispute $\mathfrak{D}$ is non-trivial, and the dispute $\mathfrak{E}$ shares the same initial action, then proposition 8 and the induction hypothesis can be applied : this works when $\mathfrak{m}$ is odd (by odd coherence) or when it is even with major bloke in X (by even coherence of the projections on X). Now if $\mathfrak{m}$ is even, with major bloke in Y, then **a** is $\xi \vdash \Upsilon$, and the odd agenda of $\mathfrak{D}''$ will contain $<\mathbf{a}; \vdash \Upsilon, \Xi>$; here we must remark that the hypothesis on **a** together with the fact that Y realizes a splitting force all even blokes in $\mathfrak{m}\lceil Y$ to be atomic : then Ξ determines the first action. Now if $[\mathfrak{m}_0, \ldots, \mathfrak{m}_q]$ is trivial or has a different first action, $<\mathbf{a}; \vdash \Upsilon, \Xi>$ will not be in the odd agenda of $\mathfrak{E}''$, and since $<\mathbf{a}>$ is the unique chronicle of length one in the two agendas, this proves odd coherence.
   In general it is enough, using case 2, to prove the odd coherence of the projections of the disputes on an arbitrary odd bloke **a** of $\mathfrak{m}\lceil Y$. **a** is obtained from odd blokes $\mathbf{b}_1, \ldots, \mathbf{b}_k$ of

$\mathfrak{m}$, and we can consider all even blokes $\mathbf{c}_1, \ldots, \mathbf{c}_l$ which shake a hand of X with some $\mathbf{b}_i$: if we take all hands occurring in some $\mathbf{c}_i$ or $\mathbf{b}_j$ and their « subblokes », we get a set Z part of an odd splitting Z, T of $\mathfrak{m}$ such that the only blokes in $Z \cap Y$ occur in $\mathbf{a}$ and whenever an even bloke meets $Z \cap X$ it is contained in it. In fact $(\mathfrak{m} \restriction Y) \restriction \mathbf{a} = (\mathfrak{m} \restriction Z) \restriction Z \cap Y$ and $(\mathfrak{m} \restriction Z) \restriction Z \cap X = \mathfrak{m} \restriction Z \cap X = (\mathfrak{m} \restriction X) \restriction Z \cap X$. From this we see that the disputes projected on Z, and the even splitting $X \cap Z, Y \cap Z$, enjoy the hypotheses of the theorem ; moreover $(\mathfrak{m} \restriction Z) \restriction Z \cap Y$ has a unique odd bloke, and we have reduced the problem to the case already treated, without increasing $\inf(m, n)$.

4. If $\mathfrak{D}'' = \mathfrak{C}''$, they are evenly coherent and since $\mathfrak{D}'$ and $\mathfrak{C}'$ are evenly coherent the two disputes are evenly coherent. So they are equal. $\square$

4.3 Designs

Definition 19

> *An* even *design, is a non-empty set $\mathcal{S}$ of disputes which are pairwise evenly coherent, and which is closed under odd restrictions : if $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p] \in \mathcal{S}$ and $q < p$ is such that $\mathfrak{m}_q$ is odd, then $[\mathfrak{m}_0, \ldots, \mathfrak{m}_q] \in \mathcal{S}$. Observe that the maul $\mathfrak{m}_0$ is common to all $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p] \in \mathcal{S}$: this is the* initial maul *of $\mathcal{S}$.*
> *Besides the parity (odd/even) of a* design, *we may consider its* polarity *(positive/negative) :*
>
> 1. *$\mathcal{S}$ is positive when its parity is the same as the parity of* $\mathfrak{m}$
> 2. *$\mathcal{S}$ is negative when its parity is opposite to the parity of* $\mathfrak{m}$

A design is a sort of strategy for the player, Even or Odd, corresponding to its parity ; styling it positive means that this player actually starts (or at least should start) ; styling it negative means that this player cannot start.

Definition 20

> *An even* design *is* winning *when its disputes are won by Even. Odd winning* designs *are defined similarly.*

The definitions make sense because even coherence is closed under odd restrictions, as well as even winning.

4.4 Orthogonality

A positive *design* and a negative *design* have at most one dispute in common, since two disputes in the intersection must be both evenly and oddly coherent ; if they actually meet, the common dispute cannot be won by both disputers, hence at most one of the *designs* can be winning.

Definition 21

> *Let $\mathcal{S}, \mathcal{T}$ be even and odd* designs *; we say that $\mathcal{S}, \mathcal{T}$ are* orthogonal *when they intersect. Given a set $\mathbf{G}$ of even* designs *we define its* orthogonal $\mathbf{G}^{\perp}$ *as the set of all odd* designs *which are orthogonal to all* designs *in $\mathbf{G}$. We define in a similar way the orthogonal of a set of odd* designs. *As usual $\mathbf{G} \subset \mathbf{G}^{\perp\perp}$ and $\mathbf{G}^{\perp\perp\perp} = \mathbf{G}^{\perp}$.*

Proposition 11

Orthogonals are never empty, namely :

1. *If* $\mathbf{G}$ *is positive, then* $\mathbf{G}^{\perp}$ *contains the* design $\mathbf{triv}^{-}$ *made of all disputes of length one and of the trivial dispute* $[\mathfrak{m}]$.
2. *If* $\mathbf{G}$ *is negative, then* $\mathbf{G}^{\perp}$ *contains the* design $\mathbf{triv}^{+}$ *reduced to the trivial dispute* $[\mathfrak{m}]$.

PROOF. — Let us assume that $\mathfrak{m}$ is even ; then :

1. All disputes in $\mathbf{triv}^{-}$ are oddly coherent, and the presence of the trivial dispute ensures closure under even restriction. Take a *design* $\mathcal{S} \in \mathbf{G}$, and a dispute in $\mathcal{S}$, then it is either of length zero, or can be restricted to a dispute of length one, still in $\mathcal{S}$ by odd restriction : in both cases it contains an element of our *design*.

2. Take a *design* $\mathcal{S}$ in $\mathbf{G}$, then it contains by even restriction the trivial dispute $[\mathfrak{m}]$, i.e. meets the *design* $\mathbf{triv}^{-}$. □

The two *designs* of our proposition are called the *trivial designs* of $\mathbf{G}^{\perp}$. Of course none of them is winning : indeed they are both based on giving up, the only difference being that $\mathbf{triv}^{+}$ directly commits suicide, whereas $\mathbf{triv}^{-}$ can only do this provided his opponent didn't commit suicide before him.

4.5 Behaviors

Definition 22

An even behavior $\mathbf{G}$ *is a set of even* designs *equal to its biorthogonal ; its orthogonal, which is a set of odd* designs *equal to its biorthogonal, is an* odd behavior*. The* extension $|\mathbf{G}|$ *of* $\mathbf{G}$ *(which is also the extension of* $\mathbf{G}^{\perp}$*) is the set of disputes which occur both in a* design *of* $\mathbf{G}$ *and a* design *of* $\mathbf{G}^{\perp}$*. The* polarity*, positive, or negative, of a behavior, is defined as for* designs*.*

We shall sometimes prefer to see a behavior as the pair $\mathbf{G}, \mathbf{G}^{\perp}$, in which case $\mathbf{G}$ (identified with $\mathbf{G}^{\perp}$) contains positive and negative designs, or even and odd designs. This may help to simplify notations and the statement of results.

Proposition 12

The extension of $\mathbf{G}$ *is closed under restriction.*

PROOF. — If $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p] \in |\mathbf{G}|$ then $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p] \in \mathcal{S} \cap \mathcal{T}$, with $\mathcal{S}, \mathcal{T}$ designs in $\mathbf{G}, \mathbf{G}^{\perp}$ respectively. If $q < p$ is such that $\mathfrak{m}_q$ is odd, then $[\mathfrak{m}_0, \ldots, \mathfrak{m}_q] \in \mathcal{S}$. Now we modify $\mathcal{T}$ as follows : we replace any dispute $[\mathfrak{n}_0, \ldots, \mathfrak{n}_i = \mathfrak{m}_q, \ldots, \mathfrak{n}_r] \in \mathcal{T}$ which is evenly coherent with $[\mathfrak{m}_0, \ldots, \mathfrak{m}_q]$ by its restriction $[\mathfrak{n}_0, \ldots, \mathfrak{n}_i]$. The set $\mathcal{T}'$ thus defined is indeed a design. Now if $\mathcal{S}' \in \mathbf{G}$, $\mathcal{S}'$ meets $\mathcal{T}$; if it happens that the meeting occurs in $\mathcal{T} - \mathcal{T}'$, then the restriction at length p of the common dispute belongs to $\mathcal{S}' \cap \mathcal{T}'$: from this $\mathcal{T}'$ belongs to $\mathbf{G}^{\perp}$ and $[\mathfrak{m}_0, \ldots, \mathfrak{m}_q] \in |\mathbf{G}|$. □

Proposition 13

Let us order designs by inclusion. Then if $\mathcal{S}$ is a subdesign of $\mathcal{T}$ and $\mathcal{S}$ is in design $\mathbf{G}$, so is $\mathcal{T}$. Conversely, any intersection of designs of $\mathbf{G}$ is still a design of $\mathbf{G}$, provided their union is a design. In particular any design $\mathcal{S}$ of $\mathbf{G}$ contains a minimum subdesign, still in $\mathbf{G}$.

PROOF. — Immediate. The minimum subdesign of $\mathcal{S}$ is indeed $\mathcal{S} \cap |\mathbf{G}|$. □

Definition 23

The extension *of $\mathcal{S}$ is defined as $\mathcal{S} \cap |\mathbf{G}|$. A design equal to its extension will be called* clean*). A* useful part *of a behavior $\mathbf{G}$ is a subset $\mathbf{G}' \subset \mathbf{G}$ such that the designs $\mathcal{T} \cup |\mathbf{G}|$; $\mathcal{T} \in \mathbf{G}'$ are exactly the clean behaviors of $\mathbf{G}$.*

Remark. — The extension of $\mathcal{S}$ is what can be « observed » if we follow the behavior. But, as in real life, designs may take into account impossible disputes (i.e. we overestimate our opponent). In practice it is much simpler to work with unclean designs : typically, if $\mathbf{G}, \mathbf{H}$ are negative, then a design for $\mathbf{G} \,\&\, \mathbf{H}$ is also a design for $\mathbf{G}$, and this will enable one to write $\mathbf{G} \,\&\, \mathbf{H}$ as $\mathbf{G} \cap \mathbf{H}$. In some sense, the fact that the extension of a design varies with the behavior is a form of *polymorphism*, analogous to the fact that a pure λ-expression may receive several types.

4.6 Embeddings

Definition 24

An embedding *is an injective map Φ from $\mathbb{H}$ to itself which preserves the relation « to be an immediate subhand », and the fact of being simple or multiple. Therefore an embedding is either* even *which means that $\Phi(\xi)$ has the parity of ξ, or* odd *which means that $\Phi(\xi)$ has a parity opposite to ξ.*

If Φ is an embedding, then it is extended in the usual way to arbitrary sets of hands, and therefore to blokes, by means of $\Phi(\Xi \vdash \Sigma) = \Phi(\Xi) \vdash \Phi(\Sigma)$. If $\mathfrak{m}$ is a maul, the maul $\Phi(\mathfrak{m})$ is defined as the tree $\{<\Phi(\sigma_0), \ldots, \Phi(\sigma_{n-1})>; <\sigma_0, \ldots, \sigma_{n-1}> \in \mathfrak{m}\}$. Finally, if $[\mathfrak{m}_0, \ldots, \mathfrak{m}_p]$ is a dispute, then the sequence $\Phi([\mathfrak{m}_0, \ldots, \mathfrak{m}_p]) = [\Phi(\mathfrak{m}_0), \ldots, \Phi(\mathfrak{m}_p)]$ is easily seen to be a dispute.
An embedding can also be applied to a *design*, and to a behavior : $\Phi(\mathbf{G}) = \mathbf{K}^{\perp\perp}$, with $\mathbf{K} = \{\Phi(\mathcal{S})\ ; \mathcal{S} \in \mathbf{G}\}$; indeed $\mathbf{K}$ is a useful part of $\Phi(\mathbf{G})$. Using embeddings, it is not difficult to define the notion of *equivalent* behaviors, and of course an even behavior may be equivalent to an odd behavior.

4.7 Basic examples

Take an *atomic* maul, consisting of $\vdash \xi$ and $\xi \vdash$; then $\mathbf{G} = \{\mathbf{triv}^+\}$ defines a positive behavior, whose orthogonal consists in all subsets of $\mathbf{triv}^-$ which are designs (i.e. which contain the trivial dispute). The behavior $\mathbf{G}$ is noted $\mathbf{0}$ (and then its orthogonal is noted $\top$) when ξ is simple, and $\mathbf{0}^\flat$ when ξ is multiple. (To be precise, we should indicate ξ in the notation, but any two choices can be related by appropriate embeddings).

5 Behavioral connectives

We shall now combine behaviors to produce new ones ; these ways will correspond to the connectives (multiplicative and additive) of linear logic. There is no particular difficulty, except that the same definition may take up to three different forms :

analytic Usual binary connectives, e.g. $\oplus, \otimes$. Since the polarity of the constituent behaviors in —say $\mathbf{G} \oplus \mathbf{G}'$— does matter, it is wiser to consider these operations as mapping positive behaviors to positive behaviors, and to add a special unary connective for the change of polarity.

synthetic Synthetic connectives ; here, given positive behaviors $\mathbf{G}_1, \ldots, \mathbf{G}_n$, we construct a new positive behavior $\mathfrak{X}\mathbf{G}_1 \ldots \mathbf{G}_n$, e.g. $\mathbf{G}_1{}^\perp \otimes (\mathbf{G}_2{}^\perp \oplus \mathbf{G}_3{}^\perp)$.

sequential Given a sequent $\vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$ or $\mathbf{G}_0 \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$ of positive behaviors, define a new behavior.

5.1 Sequents of behaviors

Let $\Xi \vdash \Sigma$ be an even bloke ; we shall (abusively) identify it with the corresponding standard maul, see example 1.

Proposition 14

projection *If $\mathcal{T}$ is an odd design with starting maul $\Xi \vdash \Sigma$, and $\eta \in \Sigma, \Xi$, then the set $\mathcal{T} \upharpoonright \eta$ of the projections on η of all disputes of $\mathcal{T}$ is an odd design with starting maul η.*

tensorization *Given odd designs $\mathcal{T}_\eta$ with starting mauls $\vdash \eta$ for all $\eta \in \Sigma, \Xi$, then the set $\bigotimes_\eta \mathcal{T}_\eta$ consisting of all disputes with starting maul $\Xi \vdash \Sigma$ whose respective projections on the η belong to $\mathcal{T}_\eta$, is an odd design.*

application *Given $\nu \in \Sigma, \Xi$, together with an even design $\mathcal{S}$ with starting maul $\Xi \vdash \Sigma$ and odd designs $\mathcal{T}_\eta$ with starting mauls $\vdash \eta$ for all $\eta \in \Sigma, \Xi$, $\eta \neq \nu$ (notation : $\mathcal{T}_\Upsilon$), then one can define the set $\mathcal{S}(\mathcal{T}_\Upsilon)$, as the set of all $\mathfrak{D} \upharpoonright \nu$ such that $\mathfrak{D} \in \mathcal{S}$ and $\mathfrak{D} \upharpoonright \eta \in \mathcal{T}_\eta$ for all $\eta \in \Sigma, \Xi$, $\eta \neq \nu$. $\mathcal{S}(\mathcal{T}_\Upsilon)$ is an even design.*

focalization $(\mathcal{S} \cap \bigotimes_\eta \mathcal{T}_\eta) \upharpoonright \nu = \mathcal{S}(\mathcal{T}_\Upsilon) \cap \mathcal{T}_\nu$.

PROOF. — The result is almost immediate from theorem 6, projection coming from 1, tensorization from 2, and application from 3. Focalization is essentially trivial. □

Definition 25

Let $\Xi \vdash \Sigma$ be an —say— even bloke, and let $\mathbf{G}_\eta$ be, for each $\eta \in \Sigma, \Xi$, a positive behavior with starting maul $\vdash \eta$; we assume that $\mathbf{G}_\eta$ is even for $\eta \in \Sigma$ and odd for $\eta \in \Xi$. We define a new behavior $\mathbf{H} = \mathbf{G}_\Xi \vdash \mathbf{G}_\Sigma$, as $\mathbf{K}^\perp$, where $\mathbf{K}$ is the set of all designs $\bigotimes_\eta \mathcal{T}_\eta$ where the $\mathcal{T}_\eta$ are odd behaviors (see definition 22) in $\mathbf{G}_\eta$ for $\eta \in \Sigma, \Xi$. This behavior is positive exactly when Ξ is empty.

The following lemma is an essential application of dog's play.

Proposition 15

Let $\nu \in \Sigma, \Xi$ and let $\mathfrak{D}$ be a dispute with initial maul $\vdash \nu$.

1. *If $\Xi = \emptyset$ or $\nu \in \Xi$ and let $\mathfrak{D}$ be a dispute with initial maul $\vdash \nu$. Then there is a dispute $I_\nu(\mathfrak{D})$ projecting as $\mathfrak{D}$ on ν and as « give up » on the $\eta \neq \nu$. The map $\mathfrak{D} \rightsquigarrow I_\nu(\mathfrak{D})$ preserves coherence, even or odd.*
2. *If $\Xi = \xi \neq \nu$, and let $\kappa \in \mathbf{A}$ (where $\mathbf{A}$ is the set of all first actions on the maul $\vdash \xi$), there is a dispute $I_{\nu,\kappa}(\mathfrak{D})$ projecting as $\mathfrak{D}$ on ν, as the trivial play on the $\eta \neq \nu$ of Σ, and as the one action dispute induced by κ on ξ. If $\mathfrak{D}, \mathfrak{D}'$ are are evenly coherent, so are $I_{\nu,\kappa}(\mathfrak{D})$ and $I_{\nu,\kappa'}(\mathfrak{D}')$.*

PROOF. —

1. Let $\Gamma = \Sigma - \nu$. Then we just follow the actions of $\mathfrak{D}$, without splitting Γ, i.e. some even blokes $\vdash \Delta$ will stay the same, some other blokes will be replaced with $\vdash \Delta, \Gamma$. In the case of an action κ that we want to « follow » in $\vdash \Delta, \Gamma$, we decide to « give » Γ to the Δ_i with the smallest index.
2. Assume that κ is induced by I and let $\Upsilon = \xi * i \, ; i \in I$, $\Gamma = \Sigma, \Upsilon - \nu$. The first action of the dispute is given by κ, then we proceed as above. □

Theorem 7

With the notations of definition 25 let $\mathbf{H} = \mathbf{G}_\Xi \vdash \mathbf{G}_\Sigma = \mathbf{K}^\perp$; then :

1. *Let us focalize on some $\nu \in \Sigma, \Xi$; then $\mathbf{H}$ consists in those even designs $\mathcal{S}$ such that for all odd designs $\mathcal{T}_\eta$ in $\mathbf{G}_\eta$ ($\eta \in \Sigma, \Xi$, $\eta \neq \nu$), $\mathcal{S}(\mathcal{T}_\Upsilon)$ is an even design of $\mathbf{G}_\nu$.*
2. *$\mathbf{K}$ is a useful part of $\mathbf{K}^{\perp\perp}$.*

PROOF. —

1. This is an immediate application of focalization : $(\mathcal{S} \cap \bigotimes_\eta \mathcal{T}_\eta) \upharpoonright \nu = \mathcal{S}(\mathcal{T}_\Upsilon) \cap \mathcal{T}_\nu$.
2. We select a focus $\nu \in \Sigma, \Xi$, and let $\mathcal{S}_\nu$ be an even design in $\mathbf{G}_\nu$. Then we define $I_\nu(\mathcal{S}_\nu) = \{I_\nu(\mathfrak{D}); \mathfrak{D} \in \mathcal{S}_\nu\}$, except if $\Xi = \xi \neq \nu$, in which case $I_\nu(\mathcal{S}_\nu) = \{I_{\nu,\kappa}(\mathfrak{D}); \mathfrak{D} \in \mathcal{S}_\nu, \kappa \in \mathbf{A}\} \cup \{[\Xi \vdash \Sigma]\})$ ($[\Xi \vdash \Sigma]$ is the trivial dispute). In both cases, $I_\nu(\mathcal{S}_\nu)$ is an even design with starting maul $\Xi \vdash \Sigma$. Choose $\bigotimes_\eta \mathcal{T}_\eta \in \mathbf{K}$, and assume that $\mathcal{S}_\nu \cap \mathcal{T}_\nu = \{\mathfrak{D}\}$:

   (a) If $\Xi = \emptyset$ or $\nu \in \Xi$, then $I_\nu(\mathcal{S}_\nu) \cap \bigotimes_\eta \mathcal{T}_\eta = \{I_\nu(\mathfrak{D})\}$.
   (b) If $\Xi = \xi \neq \nu$ and $\mathcal{T}_\xi$ is the trivial positive design, then $\bigotimes_\eta \mathcal{T}_\eta$ is trivial too and $I_\nu(\mathcal{S}_\nu) \cap \bigotimes_\eta \mathcal{T}_\eta \neq \emptyset$.
   (c) If $\Xi = \xi \neq \nu$ and $\mathcal{T}_\xi$ has a common first action κ, then $I_\nu(\mathcal{S}_\nu) \cap \bigotimes_\eta \mathcal{T}_\eta = \{I_{\nu,\kappa}(\mathfrak{D})\}$.

   This shows that $I_\nu(\mathcal{S}_\nu) \in \mathbf{H}$, and that its extension is made of three possible kinds of disputes, (a),(b),(c).
   Select $\mathcal{T} \in \mathbf{H}^\perp$. If $\mathcal{T}$ is positive and is trivial, then $\mathcal{T}$ is equal to $\bigotimes_\eta \mathcal{T}_\eta$, where the $\mathcal{T}_\eta$ are trivial. Otherwise, let $\mathcal{T}_\nu = \mathcal{T} \upharpoonright \nu$. Let ν, $\mathcal{S}_\nu$, $I_\nu(\mathcal{S}_\nu) \in \mathbf{H}$ be as above ; then $I_\nu(\mathcal{S}_\nu)$ intersects with $\mathcal{T}$. The intersection cannot correspond to a kind (b), hence it is of kind (a) or (c), i.e. it consists, depending on the polarity, in a dispute $I_\nu(\mathfrak{D})$ or $I_{\nu,\kappa}(\mathfrak{D})$. But then $\mathfrak{D} \in \mathcal{S}_\nu \cap \mathcal{T}_\nu$, and we have shown that $\mathcal{T}_\nu$ is an odd design in $\mathbf{G}_\nu$. Now, $\bigotimes_\eta \mathcal{T}_\eta \in \mathbf{K}$ and clearly contains $\mathcal{T}$, i.e. has the same extension.

□

Remark. — The basic use of the theorem is as follows : given a sequent of behaviors $\mathbf{G}_{\Xi} \vdash \mathbf{G}_{\Sigma}$, it enables one to *focalize* on some $\mathbf{G}_{\nu}$, i.e. to work with $\mathcal{S}(\mathcal{T}_{\Upsilon})$ instead of $\mathcal{S}$, and then to come back to $\mathcal{S}$.

5.2 Synthetic connectives

Definition 26

Let ξ be a simple hand, of —say— even parity. If $\mathfrak{D}$ is any dispute with an initial maul $\mathfrak{m}$ of the form $\vdash \xi * i_0, \ldots, \xi * i_k$, let $\downarrow\mathfrak{D}$ be the dispute $[\vdash \xi, \mathfrak{D}]$, to which a first action has been added. If $\mathcal{S}$ is any design with starting maul $\mathfrak{m}$, then $\downarrow\mathcal{S}$ is the set of all $\downarrow\mathfrak{D}$, when $\mathfrak{D} \in \mathcal{S}$: it is a design of the same parity and opposite polarity.

Proposition 16

If $\mathbf{G}$ is any behavior with starting maul $\mathfrak{m}$ then the set consisting of the trivial design and of all $\downarrow\mathcal{S}$, where $\mathcal{S} \in \mathbf{G}$, is a useful part of a behavior $\downarrow\mathbf{G}$, with starting maul $\vdash \xi$. $\downarrow\mathbf{G}$ has the same parity as $\mathbf{G}$ and opposite polarity. Moreover $\downarrow(\mathbf{G}^{\perp}) = (\downarrow\mathbf{G})^{\perp}$.

Proof. — Obvious. It basically consists in changing the first player, and adding a dummy action. □

Definition 27

Let ξ be a simple hand, of —say— even parity. Let $F = \{i_1, \ldots, i_n\}$ be a finite set of fingers ; for each $i \in F$ let $\mathbf{G}_i$ be an odd positive behavior with starting maul $\vdash \xi * i$. If $\mathfrak{X}$ is a synthetic connective of arity n, then we define the even positive behavior $\mathbf{H} = \mathfrak{X}\mathbf{G}_{i_1}, \ldots, \mathbf{G}_{i_n}$, with starting maul $\vdash \xi$ as $\mathbf{K}^{\perp\perp}$, where $\mathbf{K}$ consists in the following designs :

- *The trivial positive design $\mathbf{triv}^{+}$.*
- *For each maximal non-empty clique $I \in \mathfrak{X}$, for each negative (i.e. even) design $\mathcal{S} \in \vdash \mathbf{G}_{I}^{\perp}$ the design $\downarrow\mathcal{S}$. (Concretely, we add a first action, which is a splitting of ξ along I.)*

Proposition 17

1. $\mathbf{H} = \mathbf{K}$.
2. $\mathbf{H}^{\perp} = \bigcap_I \downarrow(\vdash \mathbf{G}_{I}^{\perp})$, *the intersection being taken over all maximal non-empty cliques of $\mathfrak{X}$.*

Proof. — Almost obvious ; notice the extreme simplicity of the formulation. □

5.3 Analytic connectives

If we turn our attention towards connectives like $\oplus, \otimes$, then the situation is complex, since we must take into account the polarity of the components. But there is an immediate simplification, namely to use $\downarrow$ to change polarity, so that we need to define $\mathbf{G} \oplus \mathbf{H}$ and

$\mathbf{G}\oplus\mathbf{H}$ only when $\mathbf{G}, \mathbf{H}$ are positive behaviors distinct from $\mathbf{0}^\flat$, with the same starting mauls, say $\vdash \xi$.
Then one needs two embeddings $\mathbb{F} \rightsquigarrow \mathbb{F}$, so as to make disjoint the first respective actions in the two behaviors. Once the behviors have modified in this way, then the additives take a nice form, typically $\mathbf{G}^\perp \,\&\, \mathbf{H}^\perp = \mathbf{G}^\perp \cap \mathbf{H}^\perp$, etc.
One can observe that we don't know how to define $\mathbf{G} \oplus \mathbf{0}^\flat$; of course this construction is not needed, since it corresponds to nothing in **HC**, neither in usual syntax. This is a small mystery...

5.4 Isomorphisms

Basic isomorphisms (associativity, commutativity, neutrality, distributivity, absorbtion) are easily obtained, e.g. $\mathbf{G} \otimes \mathbf{0} \simeq \mathbf{0}$. They are basically contained in proposition 17. Moreover these isomorphisms are *inner* ones, i.e. can be obtained as proofs of logical equivalence, see below. There is only one exception, namely the neutrality of $\mathbf{1}$ (and dually $\perp$) : there are definitely more designs in $\vdash \mathbf{G} \otimes \mathbf{1}, \mathbf{H}$ than in $\vdash \mathbf{G}, \mathbf{H}$. This will be fixed in the next paper, remember that $\mathbf{1}$ is an exponential.
To make things precise, we first define what is a morphism of —say— positive behaviors :

Definition 28

Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ be positive even behaviors with starting mauls $\eta * i, \eta * j, \eta * k$ (we can freely choose the simple fingers i, j, k to produce variants) ; a morphism φ from $\mathbf{G}$ to $\mathbf{H}$ is a design in $\mathbf{G} \vdash \mathbf{H}$ (beware of the following detail : in definition 25 $\mathbf{G}$ has been taken odd for questions of presentation ; in particular theorem 7 can be now stated as « $\mathbf{G} \vdash \mathbf{H}$ consists of those even designs $\mathcal{S}$ such that for all even designs $\mathcal{T}$ in $\mathbf{G}$ $\mathcal{S}\mathcal{T} \in \mathbf{H}$ ».).
Assume that $\mathbf{H}$ has been assigned $\eta * 2$ and $\eta * 4$, and let Φ be the involutive embedding which exchanges $\eta * 2$ and $\eta * 4$, i.e. $\Phi(\eta * 2 * \tau) = \eta * 4 * \tau$, $\Phi(\eta * 4 * \tau) = \eta * 2 * \tau$. Then we consider all disputes $\mathfrak{D}$ with starting maul $\eta * 2 \vdash \eta * 4$, obtained as follows : in each odd maul of the dispute, the even blokes are all of the form $\xi \vdash \Phi(\xi)$; if Odd plays using action κ, with data $\xi, I, \{\Sigma_i\}_{i \in I}$, creating $\vdash \Phi(\xi), \Xi$, then Even replies with the action $\Phi(\xi), I, \{\xi * i\}_{i \in I}$. These disputes of even length are evenly coherent. Indeed they form a morphism from $\mathbf{H}$ to itself, the identity, *noted $\iota_{\mathbf{H}}$.*
Composition $\psi\varphi$ with $\psi \in \mathbf{H} \vdash \mathbf{K}$ is defined as follows : assume that the three behaviors have been assigned starting mauls, say $\eta * 0$ for $\mathbf{G}$, $\eta * 2$ for the first $\mathbf{H}$, $\eta * 4$ for the second one, $\eta * 6$ for $\mathbf{K}$. Then $\psi\varphi$ is the set of all projections $\mathfrak{C} \lceil (\eta * 0 \vdash \eta * 6)$ which project as follows, where $\mathfrak{C}$, with starting maul
$\{\vdash \eta * 0; \eta * 0 \vdash \eta * 2; \eta * 2 \vdash \eta * 4; \eta * 4 \vdash \eta * 6; \eta * 6 \vdash\}$ which projects :

1. *As a dispute of φ on $\eta * 0 \vdash \eta * 2$.*
2. *As a dispute of ψ on $\eta * 4 \vdash \eta * 6$.*
3. *As a dispute of $\iota_{\mathbf{H}}$ on $\eta * 2 \vdash \eta * 4$.*

Proposition 18

1. *The identity is a morphism and composition maps morphisms to morphisms.*
2. *Composition is associative, and the identity is neutral on both sides.*

Proof. — Obvious. Notice that the identity is just the imitation strategy discussed in [3]. □

Theorem 8

We anticipate on next section, i.e. we assume that we know how to translate proofs into designs ; then :

1. *The identity of* $\mathbf{P} \vdash \mathbf{P}$ *is the interpretation of the* (η*-expanded version of) the identity axiom* $P \vdash P$.
2. *Composition corresponds to eager cut-elimination.*

Proof. — This result might be boring to prove, hence we quail before this task. Anyway it is is a straightforward consequence of theorems 2 and 7 and of corollary 10 (completeness). □

From this, it is easy to produce the take the canoical proofs of —say— $P \otimes (Q \oplus R) \vdash P \otimes Q \oplus P \otimes R$ and $P \otimes Q \oplus P \otimes R \vdash P \otimes (Q \oplus R)$, and to show that their interpretations φ and ψ are inverse one of the other.

So what is wrong with $\mathbf{P} \simeq \mathbf{P} \otimes \mathbf{1}$ (only for $\mathbf{P}$ positive) ? The two behaviors are isomorphic, but the isomorphism is not unique. In other terms, when Odd starts, splitting ξ into the $\xi * i$, including a multiple $\xi * j$ for $\mathbf{1}$, and then Even will react by imitation, but he must give $\xi * j$ to some $\Phi(\xi * i)$, with $i \neq j$. These various isomorphisms induce the same map, but when they are used in a functorial way, i.e. in a context, « intensionality » —to take an expression often used to compensate for the want of real structure— strikes back.

6 Adequation

We shall now look at the main results, completeness and soundness (in this order). For this we shall start with a sequent $\vdash P$ (or $P \vdash$) where P is a formula of $\mathbf{HC}$ and write the tree of its subformulas, as explained in subsection 2.5. We just pay attention to give simple fingers to usual formulas, and multiple fingers to subformulas of the form $\mathbf{0}^\flat$.

The tree is finite and we can therefore, starting with the minimal subformulas, associate a behavior $\boldsymbol{\Xi} \vdash \boldsymbol{\Sigma}$ to any sequent $\Xi \vdash \Sigma$, with starting maul $\Xi \vdash \Sigma$ (if we identify abusively a maul with a sequent).

Now, given a paraproof $\mathbf{n}$ of $\vdash P$ and a paraproof $\mathbf{m}$ of $P \vdash$, we can construct, as in subsection 2.2 a finite dispute $\mathbf{nm}$, which is indeed a dispute in the technical sense we introduced later. Now each $\mathbf{n}$ can therefore be interpreted as a design, namely the set of all $\mathbf{nm}$, when $\mathbf{m}$ varies trough the paraproofs of $P \vdash$.

6.1 Completeness

Take a sequent $\Xi \vdash \Sigma$ of $\mathbf{HC}$, and let $\boldsymbol{\Xi} \vdash \boldsymbol{\Sigma}$, be the appropriate sequent of behaviors, that we shall call $\mathbf{G}$.

Theorem 9

1. *If* $\mathbf{G}$ *is negative and* $\mathcal{S} \in \mathbf{G}$ *is clean, then :*
   (a) *If* $\Xi = \mathbf{0}^\flat$, *then* $\mathcal{S}$ *is reduced to the trivial dispute.*
   (b) *If* $\Xi = \xi = \mathfrak{X} P_1 \ldots P_n$, *let* I *be a non-empty maximal clique of* $\mathfrak{X}$, *and let* $\vdash \Sigma, \Upsilon(I)$ *be the result of the corresponding first action. Then*

$$\mathcal{S}_I = \{\mathfrak{D}; [\xi \vdash \Sigma, \mathfrak{D}] \in \mathcal{S} \text{ and the initial maul of } \mathfrak{D} \text{ is } \vdash \Sigma, \Upsilon(I)\}$$

is a design of $\vdash \boldsymbol{\Sigma}, \Upsilon(I)$. *Furthermore,* $\mathcal{S}$ *is the union of the* $\downarrow\mathcal{S}_I$ *and of the trivial dispute.*

2. *If* $\mathbf{G}$ *is positive i.e. if* $\Xi = \emptyset$, *and if* $\mathcal{S} \in \mathbf{G}$, *then :*
   (a) *Either* $\mathcal{S}$ *is the trivial design.*
   (b) *Or there is some* $\xi \in \Sigma$, *and a first action* $\xi * i, \Sigma_i \;; i \in I$ *common to all disputes of* $\mathcal{S}$, *which can therefore be written* $[\Xi \vdash \Sigma; \mathfrak{D}] \;; \mathfrak{D} \in \mathcal{T}$, *for a certain design* $\mathcal{T}$. *Moreover,* ξ *corresponds to a formula* $\mathfrak{X}P_{i_1} \ldots P_{i_n}$, *and* I *is a maximal clique in* $\mathfrak{X}$. *Finally, given* $i \in I$, *and let* $\Xi_i = \xi * i$ *; then the projection of* $\mathcal{T}$ *on* $\Xi_i \vdash \Sigma_i$ *belongs to the behavior* $\boldsymbol{\Xi}_i \vdash \boldsymbol{\Sigma}_i$.

PROOF. — We assume that the starting maul is even.

1. In case $\xi = \mathbf{0}^\flat$, then Odd cannot start, since his first action should project into an action inside the behavior $\mathbf{0}^\flat$. From this we see that the extension of $\mathcal{S}$ is reduced to the trivial dispute.
   Otherwise, one first establishes the result when $\Sigma = \emptyset$: this is exactly proposition 17, 2. Then by a back and forth focalization on ξ we get the general case.

2. By proposition 9, $\mathcal{S}$ is either reduced to the trivial dispute, or all the disputes in $\mathcal{S}$ share the same initial action, $\xi * i, \Sigma_i \;; i \in I$. In a first step we treat the case where $\Sigma = \emptyset$, which is basically definition 26 and proposition 17, 1. Then, we apply a back and forth focalization on ξ to get the general case. □

Corollary 10

Let $\mathcal{S} \in \boldsymbol{\Xi} \vdash \boldsymbol{\Sigma}$, $\mathcal{S}$ *clean. Then* $\mathcal{S}$ *is the interpretation of a paraproof of* $\mathbf{HC}$. *Moreover in case* $\mathcal{S}$ *is winning, the proof is a real one, i.e. is correct.*

PROOF. — We iterate the previous theorem, and remark that all cases correspond to a rule of $\mathbf{HC}$: 1.(a) corresponds to the axiom for $\mathbf{0}^\flat$, 1.(b) corresponds to the left rule for $\mathfrak{X}$, 2.(a) corresponds to the hypothesis, 2.(b) corresponds to one of the right rules for $\mathfrak{X}$. From this we can reconstruct a paraproof in $\mathbf{HC}$ from below, and an obvious finiteness argument makes the paraproof finite. Now in case $\mathcal{S}$ is winning, no paralogism can be used, i.e. the paraproof is correct. □

6.2 Soundness

We assume that $\Xi \vdash \Sigma$ contains exactly one formula. Observe that, due to completeness, any design $\mathcal{T}$ in $\mathbf{G}^\perp$ comes indeed from a paraproof $\mathbf{m}$ in $\mathbf{HC}$. Now, if $\mathcal{S}$ is the interpretation of a paraproof $\mathbf{n}$ of $\Xi \vdash \Sigma$, and if $\mathcal{T} \in \mathbf{G}^\perp$, then $\mathcal{S} \cap \mathcal{T}$ is nonempty : it contains $\mathbf{nm}$. This shows that $\mathcal{S} \in \mathbf{G}$. Finally, if $\mathcal{S}$ is winning, then all disputes $\mathbf{nm}$ are won by the side of $\mathbf{n}$, and from this we know that $\mathbf{n}$ is an actual proof, see subsection 2.3. We just proved the :

Theorem 11

If $\mathcal{S}$ *is the interpretation of a paraproof* $\mathbf{n}$ *of* $\Xi \vdash \Sigma$, *then* $\mathcal{S} \in \boldsymbol{\Xi} \vdash \boldsymbol{\Sigma}$ *; moreover, if* $\mathbf{n}$ *is a real proof,* $\mathcal{S}$ *is a winning design.*

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